

The Knill Graph Dimension from The Minimum Clique Cover

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ABSTRACT

In this paper we prove that the recursive (Knill) dimension of the join of two graphs has a simple formula in terms of the dimensions of the component graphs: $\dim(G_1 + G_2) = 1 + \dim G_1 + \dim G_2$. We use this formula to derive an expression for the Knill dimension of a graph from its minimum clique cover. A corollary of the formula is that a graph made of the arbitrary union of complete graphs K_N of the same order N will have dimension $N - 1$. We finish by finding lower and upper bounds on the Knill dimension of a graph in terms of its clique number.

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1. INTRODUCTION

A purely graph-theoretical definition of the dimension of a simple graph was given by Oliver Knill in [3] and explored in greater depth in [4]. The dimension of a finite simple graph G is given recursively as 1 plus the arithmetic mean of the dimension of the unit spheres at each vertex v .

$$\begin{aligned} \dim G &= \frac{1}{|V(G)|} \sum_v \dim_G(v) \\ \dim_G(v) &= 1 + \sum_{v' \in N(v)} \dim S_G(v'), \end{aligned} \tag{1.1}$$

where $V(G)$ is the set of vertices in the graph, $N(v)$ is the set of vertices connected to v , i.e., the neighbors of v , and $S_G(v)$ is the unit sphere at v defined as the induced subgraph in G whose vertex set is $N(v)$. In this paper we will refer to this recursive definition of the graph dimension as the *Knill* dimension.

In [5], the Knill dimension is shown to obey

$$\dim (S^k + S^l) = 1 + k + l,$$

where S^k is a k -dimensional geometric sphere, defined recursively as a graph such that every unit sphere in the graph is a $(k - 1)$ -dimensional geometric sphere (and likewise for S^l), and $S^k + S^l$ is the join graph of S^k and S^l . We will show that this formula generalizes to the join of any graphs, i.e., that for any two graphs G_1 and G_2 , the Knill dimension of the join of the two graphs is

$$\dim (G_1 + G_2) = 1 + \dim G_1 + \dim G_2. \tag{1.2}$$

This general formula will be used to derive an expression for the Knill dimension of a graph from its minimum clique cover, and to place bounds on the Knill dimension.

One main benefit of (1.2) is that computing the Knill dimension recursively from the definition is very costly for dense graphs as the number of computations needed grows factorially with the number of vertices. On the other hand, dense graphs are likely to be the join graphs of two or more disconnected graphs. The dimension sum formula shown above can speed up the calculation of the Knill significantly for densely connected graphs.

We begin section 2 with a review of the graph theory definitions relevant to our discussion. We introduce a few non-standard notations for convenience, such as K_V for the complete graph formed by connecting each vertex in the set V by an edge, and the minimum and maximum clique number of a graph. For more in-depth introduction of graph theory refer to [1] and [2]. Section 3 gives the proof of (1.2). In section 4 we derive a formula for the Knill dimension of a graph from the minimum clique cover. In section 5 we place some bounds on the Knill dimension.

2. GRAPH THEORY DEFINITIONS

Definition 2.1. A **simple graph** $G(V, E)$ is a set of vertices (or nodes) V together with a set of edges E whose members are pairs of nodes. The cardinality of the vertex set $|V|$ of the graph is called the **order** of the graph. It is standard to simply write $|G|$ to refer to the order of a graph, and we will do so hence forth.

For general graphs the edges can be directed or undirected, and so the pairs of nodes in the edge space may be ordered or unordered. For undirected simple graphs the order is immaterial. Fig. 1 below gives an example of a simple undirected graph whose vertex set is $V = \{1, 2, 3, 4, 5, 6\}$ and edge set is $E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}$.

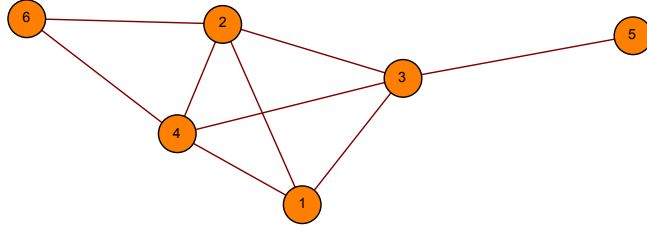


FIG. 1: A graph with 6 nodes.

Definition 2.2. The **complete graph** on N vertices, denoted by K_N is a graph where every one of the N vertices is connected with every other vertex. It has $N(N - 1)/2$ edges. Here N refers to the cardinality of the vertex set.

If V is a set of vertices, we define K_V to be the complete graph whose vertex set is V . For example, $K_{\{1,2,3\}}$ is the complete graph with vertex set $V = \{1, 2, 3\}$.

Definition 2.3. A **subgraph** of a graph is another graph whose vertex set and edge set are subsets of the vertex and edge sets of the graph.

$$G'(V', E') \subseteq G(V, E) \implies V' \subseteq V \text{ \& } E' \subseteq E.$$

Definition 2.4. A **induced subgraph** of a graph G is a subgraph whose vertex set V' is a subset of the vertex set of G and whose edge set is made up of all edges in G that connect the vertices in V' . The induced subgraph of G over V' is denoted by $G[V']$. If H is a subgraph of G with vertex set $V(H)$, we will denote the induced subgraph in G over $V(H)$ simply as $G[H]$ instead of writing $G[V(H)]$.

If H is a subgraph of G with vertex set $V' \subseteq V$, it is common to denote by $G - H$ the induced graph $G[V \setminus V']$. Similarly if $v_0 \in V$ is a vertex in G , we write $G - v_0$ for the induced subgraph $G[V \setminus v_0]$.

Example 2.1. The graph with vertex set $V' = \{1, 2, 3, 4\}$ will be induced subgraph

of G shown in Fig. 1 if its edge set is $\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. The graph with the same vertex set $\{1, 2, 3, 4\}$ but with edge set $\{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ is a subgraph of G , but not induced subgraph because it doesn't include the edges $\{\{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

Definition 2.5. A **clique** is an induced subgraph that is also a complete graph.

A clique is a **maximal clique** if adding any other vertex $u \in V(G)$ to the vertices in the clique renders the resulting induced subgraph not a complete graph. In other words, a clique is maximal if it can not be made a larger clique by including more vertices.

The order of the biggest maximal clique of a graph G is called the **clique number** of the graph and denoted by $\omega(G)$. This is a standard definition and notation of the clique number of a graph in the literature. However, we will find it convenient to separate the clique number in two; the **maximum clique number** referring to the order of the largest maximal clique (denoted by $\omega(G)$), and the **minimum clique number** referring to the order of the smallest maximal clique. There is no standard notation in the literature for the minimum clique number, so we will use the notation $\gamma(G)$ to denote the order of the smallest maximal clique in G .

Definition 2.6. **Clique cover** of a graph G is a set of cliques whose union covers G , i.e., the union of the vertex set of the cliques is equal to the vertex set of G , and the union of the edge sets of the cliques is the edge set of G . The **minimum clique cover** of a graph uses the fewest possible cliques to cover the graph. Therefore, the cliques in the minimum clique cover are all maximal cliques. The number of cliques in the minimum clique cover is called the **clique cover number**, and denoted by $\theta(G)$.

There are three distinct notions of clique cover. The **vertex clique cover** is a set of cliques whose union covers the vertices of G , but not necessarily the edges. Therefore, the vertex clique cover is a partition of the vertex set of the graph into

cliques. The vertex clique cover number is then the minimum number of cliques needed to cover the vertices of the graph. The cliques in the vertex clique cover need not be maximal since the union of the edge sets of the cliques need not cover the edge set of the graph. On the other hand, the **edge clique cover** of a graph is a set of cliques that cover the edges of the graph; i.e, the union of the edge sets of the cliques in the edge clique cover is equal to the edge set of the graph. The minimum edge clique cover uses the fewest possible number of cliques, so the cliques in the minimum edge clique cover are maximal cliques. The minimum number of cliques needed to form an edge clique cover is called the edge clique cover number, denoted by $cc(G)$. However, the edge clique cover need not cover all vertices of the graph; for example, isolated vertices of the graph are not contained by any clique in the edge clique cover.

In this paper we will be concerned with clique cover of both vertices and edges of the graph. We will refer by the clique cover (without qualifying vertex or edge) to the set of cliques which cover both the vertices and edges of the graph; i.e., the union of the vertex sets of the clique cover equals the vertex set of the graph, and the union of the edge set of the clique cover equals the edge set of the graph. The vertex clique cover number is equal to the clique cover number, even though the cliques in the minimum vertex clique cover are not necessarily maximal. The vertex clique cover number will equal the edge clique cover number for connected graphs.

Example 2.2. The following induced subgraphs of G in Fig. 1 are all cliques in G : $G[\{1, 2\}]$, $G[\{1, 2, 3\}]$, $G[\{1, 2, 3, 4\}]$, $G[\{2, 4, 6\}]$, $G[\{3, 5\}]$. But the maximal cliques are only the induced subgraphs $G[\{1, 2, 3, 4\}] = K_{\{1,2,3,4\}}$, $G[\{2, 4, 6\}] = K_{\{2,4,6\}}$, $G[\{3, 5\}] = K_{\{3,5\}}$. The set of these maximal cliques forms the minimum clique cover of G , so the clique cover number of the graph is 3. The cliques $K_{\{6\}}$, $K_{\{1,2,3,4\}}$, $K_{\{5\}}$ together form a vertex clique cover of G .

Example 2.3. The maximal cliques of the graph shown in Fig. 1 are highlighted in the figure below.

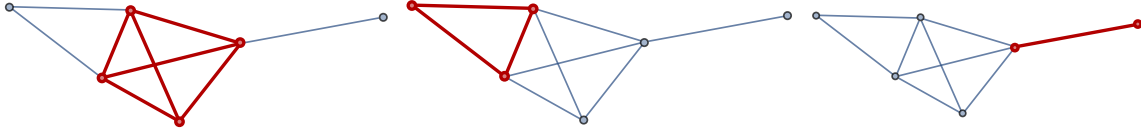


FIG. 2: The maximal cliques in the graph of Fig. 1.

Definition 2.7. The **degree** of a vertex in a graph G , denoted by $\deg_G(v)$ or simply $\deg(v)$ if the graph being referred to is obvious, is the number of vertices connected to the vertex by an edge. If the degree of a vertex is zero, it is said to be an **isolated** vertex.

Definition 2.8. The **distance** $d(v_1, v_2)$ between two vertices v_1 and v_2 is defined as the length of the shortest path connecting v_1 and v_2 . If $\{\ell_p(v_1, v_2)\}$ is the set of the lengths of all paths between v_1 and v_2 , then

$$d(v_1, v_2) = \min\{\ell_p(v_1, v_2)\}.$$

Definition 2.9. A **sphere** in G of radius r centered at the vertex v , denoted by $S_G(v, r)$ is an induced subgraph of G whose vertices are all vertices in G with distance of exactly r from v .

The unit sphere will have $r = 1$. We will abbreviate the notation of the unit sphere at v and write simply $S_G(v)$ without the 1. The radius is assumed to be one unless explicitly stated. The vertices of the unit sphere at v are called the **neighbors** of v . The degree of a vertex is therefore equal to the order of the unit sphere at the vertex.

Definition 2.10. A **ball** in G of radius r centered at v is denoted by $B_G(v, r)$. It is the induced subgraph whose vertices are all vertices in G at a distance of r or less from v . We write simply $B_G(v)$ for the unit ball with $r = 1$.

Definition 2.11. The **union** of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is $G_1 \cup G_2 = G(V_1 \cup V_2, E_1 \cup E_2)$. The intersection of the two graphs is $G_1 \cap G_2 = G(V_1 \cap V_2, E_1 \cap E_2)$.

Definition 2.12. The **Join** of two disjoint graphs $G_1(V_1, E_1), G_2(V_2, E_2)$, denoted by $G_1 + G_2$, is another graph $G(V, E)$ whose vertex set is $V = V_1 \cup V_2$ and whose edge set is $E = E_1 \cup E_2 \cup \{\{v_1, v_2\} \mid \forall v_1 \in V_1, v_2 \in V_2\}$, i.e. the union of the edge sets E_1, E_2 , and all edges formed by linking every vertex in G_1 with every vertex in G_2 . Since the component graphs of the join are disjoint, the order of the join graph is the sum of the order of the component graphs, i.e., $|G_1 + G_2| = |G_1| + |G_2|$.

Example 2.4. The unit ball $B_G(v)$ and the unit sphere $S_G(v)$ are related by

$$B_G(v) = v + S_G(v)$$

Definition 2.13. The **Knill dimension** of a graph, denoted by $\dim(G)$ is defined recursively as

$$\dim(G) = \begin{cases} -1, & \text{if } G \text{ is the empty graph} \\ \frac{1}{|G|} \sum_v \dim_G(v) & \text{otherwise} \end{cases}$$

$$\dim_G(v) = 1 + \dim S_G(v) \tag{2.1}$$

Example 2.5. The completely disconnected graph over N nodes has dimension of 0.

Example 2.6. The complete graph over N nodes K_N has dimension of $N - 1$. The Knill dimension matches with the Euclidean dimension of simplexes, if the complete graph K_N is thought of as a $(N - 1)$ -simplex embedded in N -dimensional Euclidean space.

Example 2.7. All bipartite graphs are 1-dimensional. That is because $S_G(v)$ is completely isolated graph for any v in a bipartite graph; which means $\dim_G(v) = 1 + \dim S_G(v) = 1$, and $\dim G = \frac{1}{|G|} \sum_v 1 = 1$.

Example 2.8. All tree graphs are 1-dimensional since all tree graphs are bipartite.

Example 2.9. All Cycle graphs C_n are 1-dimensional for $n > 3$.

3. DIMENSION OF THE JOIN OF GRAPHS

One immediate observation of the Knill dimension given in [4] is that the dimension of a disconnected graph with two components is the weighted average of the dimensions of the components. This is true for any disconnected graphs with arbitrary number of components. We will restate this observation in its generality in the lemma below.

Lemma 3.1. *The dimension of a disconnected graph is the weighted average of the dimensions of the components, where the weight of a component is the ratio of the order of the component to order of the graph. Let $G = G_1 \cup G_2 \cup \dots \cup G_N$, where the G_i are components of G , then*

$$\dim G = \sum_{i=1}^N \frac{|G_i|}{|G|} \dim G_i. \quad (3.1)$$

Proof: We begin by observing that for a vertex $v \in G_i$, $S_G(v) = S_{G_i}(v)$, therefore $\dim_G(v) = \dim_{G_i}(v)$. Then,

$$\begin{aligned} |G| \dim G &= \sum_{v \in G} \dim_G(v) \\ &= \sum_{i=1}^N \sum_{v \in G_i} \dim_G(v) \\ &= \sum_{i=1}^N \sum_{v \in G_i} \dim_{G_i}(v) \\ &= \sum_{i=1}^N |G_i| \dim G_i \end{aligned}$$

Lemma 3.2. *The dimension of the join of two arbitrary graphs is one plus the sum of the dimensions of each; i.e.,*

$$\dim(G_1 + G_2) = 1 + \dim G_1 + \dim G_2. \quad (3.2)$$

Proof: This lemma generalizes a corollary 7 of [4], which states the join of a k -dimensional and an ℓ -dimensional geometric spheres is a geometric sphere of dimension $n = k + \ell + 1$. We will prove the theorem by induction over the order of $G_1 + G_2$. The base case is when either $|G_1| = 0$ or $|G_2| = 0$. Since the dimension of the empty graph is by definition -1 , we have

$$\dim(G_1 + \emptyset) = 1 + \dim G_1 + \dim \emptyset = \dim G_1.$$

Assume the formula is true for arbitrary $|G_1 + G_2| = |G_1| + |G_2| = k$ and neither graph is empty, we will show that the formula holds when the order of the join is $k + 1$. Let's call the join $G_1 + G_2 = G$ with $|G| = |G_1| + |G_2| = k + 1$. First we observe that if a vertex $v \in G_1$, then $S_G(v) = S_{G_1}(v) + G_2$ and similarly if $v \in G_2$, then $S_G(v) = S_{G_2}(v) + G_1$ (See Fig. 3 below). Then,

$$\begin{aligned} |G|\dim(G) &= \sum_{v \in G_1} \dim_G(v) + \sum_{v \in G_2} \dim_G(v) \\ (k+1)\dim(G) &= \sum_{v \in G_1} (1 + \dim S_G(v)) + \sum_{v \in G_2} (1 + \dim S_G(v)) \\ &= \sum_{v \in G_1} \left(1 + \dim(S_{G_1}(v) + G_2)\right) + \sum_{v \in G_2} \left(1 + \dim(G_1 + S_{G_2}(v))\right) \\ &= \sum_{v \in G_1} \left(1 + (1 + \dim S_{G_1}(v) + \dim G_2)\right) + \sum_{v \in G_2} \left(1 + (1 + \dim G_1 + \dim S_{G_2}(v))\right) \\ &= |G_1| + |G_1|\dim G_1 + |G_1|\dim G_2 + |G_2| + |G_2|\dim G_1 + |G_2|\dim G_2 \\ &= (|G_1| + |G_2|)(1 + \dim G_1 + \dim G_2) \\ \dim(G_1 + G_2) &= 1 + \dim G_1 + \dim G_2 \end{aligned}$$

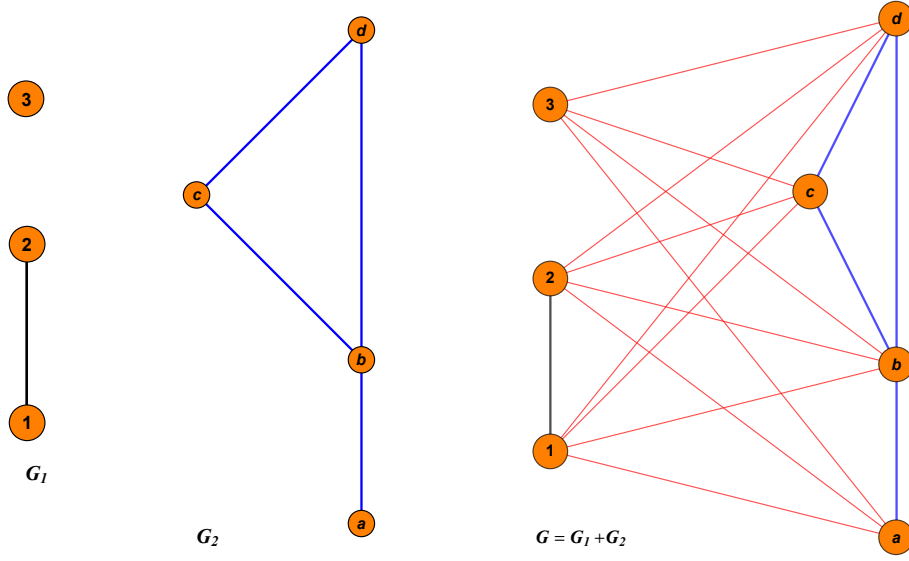


FIG. 3: Two graphs G_1 and G_2 are shown on the left. Their join is on the right. For any vertex in G_1 , the sphere in G at v is the join of the sphere in G_1 at v and G_2 . The same holds for any vertex $v \in G_2 \implies S_G(v) = S_{G_2}(v) + G_1$.

In the fourth line we applied the inductive assumption that the formula holds when the order of the join graph is k , which applies here since $|S_{G_1}(v) + G_2| \leq k$, and similarly $|G_1 + S_{G_2}(v)| \leq k$ for any v .

Corollary 3.1. *If $G = G_1 + G_2 + \dots + G_k$, then $\dim G = (k - 1) + \dim G_1 + \dim G_2 + \dots + \dim G_k$.*

Proof:

$$\begin{aligned}
 \dim G &= \dim \left(G_1 + (G_2 + \dots + G_k) \right) \\
 &= 1 + \dim G_1 + \dim (G_2 + G_3 + \dots + G_k) \\
 &= 2 + \dim G_1 + \dim G_2 + \dim (G_3 + G_4 + \dots + G_k) \\
 &\vdots \\
 &= (k - 1) + \dim G_1 + \dim G_2 + \dots + \dim G_k
 \end{aligned} \tag{3.3}$$

Example 3.1. The complete graph K_N is the join of N copies of K_1 . So, $\dim K_N = \dim (K_1 + K_1 + \cdots + K_1) = (N - 1) + N \dim K_1 = N - 1$.

Corollary 3.2. *The dimension of any vertex in G is the dimension of the unit ball at that vertex.*

$$\dim_G(v) = \dim B_G(v) = 1 + \dim S_G(v). \quad (3.4)$$

Proof: This follows immediately from the fact that $B_G(v) = v + S_G(v)$ so that $\dim_G(v) = 1 + \dim S_G(v) = \dim (v + S_G(v)) = \dim B_G(v)$.

4. KNILL DIMENSION FROM THE MINIMUM CLIQUE COVER

In this section, let us assume that the minimum clique cover of the graph G is known. Let m be the clique cover number, and let $K_{V_1}, K_{V_2}, \dots, K_{V_m}$ be the maximal cliques in the minimum clique cover with vertex sets V_1, V_2, \dots, V_m . Let us introduce some notations that will simplify the discussion. We will use the notation $K_{V_i V_j}$ to denote $G[V_i \cap V_j]$, the induced graph in G over the vertices in the intersection of K_{V_i} and K_{V_j} . By extension, we will use $K_{V_i V_j V_k} = G[V_i \cap V_j \cap V_k]$, etc. Since the intersection of two complete graphs is another complete graph, each of the $K_{V_i V_j}, K_{V_i V_j V_k}, \dots$ is also a complete graph. We will also use $G[K_{V_i} \cup K_{V_j}]$ to refer to the induced graph $G[V_i \cup V_j]$ over the union of the vertex sets of K_{V_i} and K_{V_j} .

It will be useful to define a notation for the number of vertices in a given maximal clique that are not contained in any of the other maximal cliques in the clique cover. Let $||K_{V_i}||$ be the number of vertices in K_{V_i} that are not in any of the other cliques. Similarly, let $||K_{V_i V_j}||$ be the number of vertices contained in the intersection $K_{V_i V_j}$ but not in any of the other maximal cliques, etc. Then, by the inclusion-exclusion principle,

$$\begin{aligned}
\|K_{V_i}\| &= |K_{V_i}| - \sum_{\substack{j \neq i \\ j=1}}^m |K_{V_i V_j}| + \sum_{\substack{j, k \neq i \\ j < k=1}}^m |K_{V_i V_j V_k}| - \cdots \pm |K_{V_1 V_2 \dots V_m}| \\
\|K_{V_i V_j}\| &= |K_{V_i V_j}| - \sum_{\substack{k \neq i, j \\ k=1}}^m |K_{V_i V_j V_k}| + \sum_{\substack{k_1, k_2 \neq i, j \\ k_1 < k_2=1}}^m |K_{V_i V_j V_{k_1} V_{k_2}}| - \cdots \pm |K_{V_1 V_2 \dots V_m}| \\
&\vdots \\
\|K_{V_1 V_2 \dots V_m}\| &= |K_{V_1 V_2 \dots V_m}|
\end{aligned} \tag{4.1}$$

Therefore,

$$\begin{aligned}
|G| &= \sum_{i=1}^m \|K_{V_i}\| + \sum_{i > j=1}^m \|K_{V_i V_j}\| + \sum_{i_1 > i_2 > i_3=1}^m \|K_{V_{i_1} V_{i_2} V_{i_3}}\| + \cdots \\
&\quad + \sum_{i_1 > i_2 > \dots > i_{m-1}=1}^m \|K_{V_{i_1} V_{i_2} \dots V_{i_{m-1}}}\| \\
&\quad + \|K_{V_1 V_2 \dots V_m}\|
\end{aligned}$$

Lemma 4.1. *Let G be a graph with clique cover number of 2 so that $G = K_{V_1} \cup K_{V_2}$.*

Then,

$$\left(|G| - |K_{V_1 V_2}|\right) \dim G = \|K_{V_1}\| \dim K_{V_1} + \|K_{V_2}\| \dim K_{V_2} \tag{4.2}$$

Proof: If we remove the $|K_{V_1 V_2}|$ vertices in G we end up with a disconnected graph $G^0 = G - K_{V_1 V_2}$ with two complete graph components, $K_{V_1 \setminus (V_1 \cap V_2)}$ and $K_{V_2 \setminus (V_1 \cap V_2)}$. The order of the resulting graph is $|G^0| = |G| - |V_{V_1 V_2}| = \|K_{V_1}\| + \|K_{V_2}\|$. Using lemma 3.1,

$$\begin{aligned}
|G^0| \dim G^0 &= |K_{V_1 \setminus (V_1 \cap V_2)}| \dim (K_{V_1 \setminus (V_1 \cap V_2)}) + |K_{V_2 \setminus (V_1 \cap V_2)}| \dim (K_{V_2 \setminus (V_1 \cap V_2)}) \\
&= \|K_{V_1}\| (|K_{V_1}| - |K_{V_1 V_2}|) + \|K_{V_2}\| (|K_{V_2}| - |K_{V_1 V_2}|)
\end{aligned}$$

$$\begin{aligned}
|G^0|\dim G^0 &= \|K_{V_1}\|\dim K_{V_1} + \|K_{V_2}\|\dim K_{V_2} - \dim K_{V_1V_2}(\|K_{V_1}\| + \|K_{V_2}\|) \\
&= \|K_{V_1}\|\dim K_{V_1} + \|K_{V_2}\|\dim K_{V_2} - |G^0|\dim K_{V_1V_2}
\end{aligned} \tag{4.3}$$

Then, since $G = G^0 + K_{V_1V_2}$, using lemma 3.2,

$$\begin{aligned}
\dim G &= 1 + \dim G^0 + \dim K_{V_1V_2} \\
&= \dim G^0 + |K_{V_1V_2}| \\
|G^0|\dim G &= |G^0|\dim G^0 + |G^0||K_{V_1V_2}| \\
&= \|K_{V_1}\|\dim K_{V_1} + \|K_{V_2}\|\dim K_{V_2} \\
\left(|G| - |K_{V_1V_2}|\right)\dim G &= \|K_{V_1}\|\dim K_{V_1} + \|K_{V_2}\|\dim K_{V_2}
\end{aligned}$$

Theorem 4.1. *Let $\{K_{V_1}, K_{V_2}, \dots, K_{V_m}\}$ be the set of maximal cliques in the minimum clique cover of G and let $K_L = \bigcap_{i=1}^m K_{V_i}$. Further, define the set M to be $M = \{1, 2, \dots, m\}$. Then,*

$$\begin{aligned}
\left(|G| - |K_L|\right)\dim G &= \sum_{i \in M} \|K_{V_i}\| \dim K_{V_i} \\
&+ \sum_{\{i,j\} \subset M} \|K_{V_iV_j}\| \dim G[K_{V_i} \cup K_{V_j}] \\
&+ \sum_{\{i_1, i_2, i_3\} \subset M} \|K_{V_{i_1}V_{i_2}V_{i_3}}\| \dim G[K_{V_{i_1}} \cup K_{V_{i_2}} \cup K_{V_{i_3}}] \\
&+ \vdots \\
&+ \sum_{\{i_1, i_2, \dots, i_{m-1}\} \subset M} \|K_{V_{i_1}V_{i_2}\dots V_{i_{m-1}}}\| \dim G\left[\bigcup_{i \in \{i_1, i_2, \dots, i_{m-1}\}} K_{V_i}\right].
\end{aligned} \tag{4.4}$$

Here, the sums $\sum_{\{i,j\} \subset M}$, $\sum_{\{i_1, i_2, i_3\} \subset M}$, etc. refers to the sums over all subsets of M of cardinality 2, 3, etc. The theorem is stating that to find the dimension of a graph with m cliques in the minimum clique cover, we can proceed vertex by vertex, first selecting all vertices that are contained in only one clique, whose dimension in G is simply the dimension of the clique that contains them, then those vertices in the

intersection of only two cliques, etc.

Proof: We will do inductive proof on the clique number m . The case when $m = 1$ is trivial. The case $m = 2$ is proven in lemma 4.1. Before doing the inductive step, it is instructive to prove the case $m = 3$. Let $K_{V_1}, K_{V_2}, K_{V_3}$ be the three maximal cliques in G .

First let's assume that $|K_L| = 0$, i.e., there are no vertices in the intersection of all three maximal cliques, then,

$$\begin{aligned}
|G|\dim G &= \sum_{v \in K_{V_1} \setminus (K_{V_2} \cup K_{V_3})} \dim_G(v) + \sum_{v \in K_{V_2} \setminus (K_{V_1} \cup K_{V_3})} \dim_G(v) + \sum_{v \in K_{V_3} \setminus (K_{V_1} \cup K_{V_2})} \dim_G(v) \\
&+ \sum_{v \in K_{V_1 V_2} \setminus K_{V_3}} \dim_G(v) + \sum_{v \in K_{V_1 V_3} \setminus K_{V_2}} \dim_G(v) + \sum_{v \in K_{V_2 V_3} \setminus K_{V_1}} \dim_G(v) \\
&= ||K_{V_1}|| \dim K_{V_1} + ||K_{V_2}|| \dim K_{V_2} + ||K_{V_3}|| \dim K_{V_3} \\
&+ ||K_{V_1 V_2}|| \dim G[K_{V_1} \cup K_{V_2}] + ||K_{V_1 V_3}|| \dim G[K_{V_1} \cup K_{V_3}] \\
&+ ||K_{V_2 V_3}|| \dim G[K_{V_2} \cup K_{V_3}],
\end{aligned}$$

as desired. We have used the fact that for any vertex v in $(K_{V_1} \cap K_{V_2}) \setminus K_{V_3}$, $B_G(v) = G[K_{V_1} \cup K_{V_2}]$ and so by corollary 3.2 $\dim_G(v) = \dim B_G(v) = \dim G[K_{V_1} \cup K_{V_2}]$.

For the case of $|K_L| > 0$, let us denote by G^0 the graph $G - K_L$, so that when $|K_L| = 0$, we have $G = G^0$. Then, the maximal cliques of G^0 are simply the maximal cliques of G with the $|K_L|$ nodes in K_L removed from each of them; i.e., if K_{V_i} is a maximal clique in G , then $K_{V_i \setminus V(K_L)} = K_{V_i} - K_L$ is a maximal clique in G^0 .

Further, note that the numbers $||K_{V_i}||, ||K_{V_i V_j}||, \dots$ are the same in G^0 and G since

the vertices removed are contained in the intersection of all maximal cliques. Then,

$$\begin{aligned}
|G^0| \dim G^0 &= ||K_{V_1}|| \dim (K_{V_1} - K_L) + ||K_{V_2}|| \dim (K_{V_2} - K_L) + ||K_{V_3}|| \dim (K_{V_3} - K_L) \\
&\quad + ||K_{V_1 V_2}|| \dim G[(K_{V_1} - K_L) \cup (K_{V_2} - K_L)] \\
&\quad + ||K_{V_1 V_3}|| \dim G[(K_{V_1} - K_L) \cup (K_{V_3} - K_L)] \\
&\quad + ||K_{V_2 V_3}|| \dim G[(K_{V_2} - K_L) \cup (K_{V_3} - K_L)]
\end{aligned}$$

But, by lemma 3.2,

$$\begin{aligned}
\dim (K_{V_i} - K_L) &= \dim K_{V_i} - |K_L|, \quad \text{and} \\
\dim G[(K_{V_i} - K_L) \cup (K_{V_j} - K_L)] &= \dim G[(K_{V_i} \cup K_{V_j}) - K_L] \\
&= \dim G[K_{V_i} \cup K_{V_j}] - |K_L|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|G^0| \dim G^0 &= ||K_{V_1}|| \dim K_{V_1} + ||K_{V_2}|| \dim K_{V_2} + ||K_{V_3}|| \dim K_{V_3} \\
&\quad + ||K_{V_1 V_2}|| \dim G[K_{V_1} \cup K_{V_2}] + ||K_{V_1 V_3}|| \dim G[K_{V_1} \cup K_{V_3}] \\
&\quad + ||K_{V_2 V_3}|| \dim G[K_{V_2} \cup K_{V_3}] \\
&\quad - |K_L| \left(||K_{V_1}|| + ||K_{V_2}|| + ||K_{V_3}|| + ||K_{V_1 V_2}|| + ||K_{V_1 V_3}|| + ||K_{V_2 V_3}|| \right) \\
&= ||K_{V_1}|| \dim K_{V_1} + ||K_{V_2}|| \dim K_{V_2} + ||K_{V_3}|| \dim K_{V_3} \\
&\quad + ||K_{V_1 V_2}|| \dim G[K_{V_1} \cup K_{V_2}] + ||K_{V_1 V_3}|| \dim G[K_{V_1} \cup K_{V_3}] \\
&\quad + ||K_{V_2 V_3}|| \dim G[K_{V_2} \cup K_{V_3}] - |K_L| |G^0|
\end{aligned} \tag{4.5}$$

In the last line we have used $|G^0| = ||K_{V_1}|| + ||K_{V_2}|| + ||K_{V_3}|| + ||K_{V_1 V_2}|| + ||K_{V_1 V_3}|| + ||K_{V_2 V_3}||$.

Then, since $G = G^0 + K_L$, we have $\dim G = \dim G^0 + |K_L|$, and

$$\begin{aligned} |G^0| \dim G &= |G^0| \dim G^0 + |G^0| |K_L| \\ (|G| - |K_L|) \dim G &= ||K_{V_1}|| \dim K_{V_1} + ||K_{V_2}|| \dim K_{V_2} + ||K_{V_3}|| \dim K_{V_3} \\ &\quad + ||K_{V_1 V_2}|| \dim G[K_{V_1} \cup K_{V_2}] + ||K_{V_1 V_3}|| \dim G[K_{V_1} \cup K_{V_3}] \\ &\quad + ||K_{V_2 V_3}|| \dim G[K_{V_2} \cup K_{V_3}] \end{aligned}$$

which proves the case for $m = 3$.

We now proceed with the inductive step for the proof of the formula for the general clique cover number. For the inductive step, assume the formula in theorem 4.1 holds true for m maximal cliques in the minimum clique cover of G . For the case when G has $m + 1$ maximal cliques, as before let $K_L = \bigcap_{i=1}^{m+1} K_{V_i}$, and let $G^0 = G - K_L$. Then,

$$\begin{aligned} |G^0| \dim G^0 &= \sum_{i=1}^{m+1} ||K_{V_i}|| \dim G[K_{V_i} - K_L] \tag{4.6} \\ &+ \sum_{i>j=1}^{m+1} ||K_{V_i V_j}|| \dim G[(K_{V_i} - K_L) \cup (K_{V_j} - K_L)] \\ &+ \sum_{i>j>k=1}^{m+1} ||K_{V_i V_j V_k}|| \dim G[(K_{V_i} - K_L) \cup (K_{V_j} - K_L) \cup (K_{V_k} - K_L)] \\ &+ \vdots \\ &+ \sum_{i_1>i_2>\dots>i_m=1}^{m+1} ||K_{V_{i_1} V_{i_2} \dots V_{i_m}}|| \dim G[\bigcup_{i \in \{i_1, i_2, \dots, i_m\}} (K_{V_i} - K_L)] \end{aligned}$$

For of each of the induced subgraphs, we have

$$\begin{aligned} \dim G[(K_{V_{i_1}} - K_L) \cup \dots \cup (K_{V_{i_k}} - K_L)] &= \dim G[(K_{V_{i_1}} \cup \dots \cup K_{V_{i_k}}) - K_L] \\ &= \dim G[K_{V_{i_1}} \cup \dots \cup K_{V_{i_k}}] - |K_L|. \end{aligned}$$

Therefore,

$$\begin{aligned}
|G^0| \dim G^0 &= \left(\sum_{i=1}^{m+1} \|K_{V_i}\| \dim G[K_{V_i}] \right. \\
&+ \sum_{i>j=1}^{m+1} \|K_{V_i V_j}\| \dim G[K_{V_i} \cup K_{V_j}] \\
&+ \vdots \\
&+ \left. \sum_{i_1>i_2>\dots>i_m=1}^{m+1} \|K_{V_{i_1} V_{i_2} \dots V_{i_m}}\| \dim G\left[\bigcup_{i \in \{i_1, i_2, \dots, i_m\}} K_{V_i} \right] \right) - |K_L| |G^0|
\end{aligned} \tag{4.7}$$

In the last line we have used (4.2).

Coming back to G , since $G = G^0 + K_L$, we again have $\dim G = \dim G^0 + |K_L|$, and

$$\begin{aligned}
|G^0| \dim G &= |G^0| \dim G^0 + |G^0| |K_L| \\
&= \sum_{i=1}^m \|K_{V_i}\| \dim G[K_{V_i}] \\
&+ \sum_{i>j=1}^m \|K_{V_i V_j}\| \dim G[K_{V_i} \cup K_{V_j}] \\
&+ \sum_{i>j>k=1}^m \|K_{V_i V_j V_k}\| \dim G[K_{V_i} \cup K_{V_j} \cup K_{V_k}] \\
&+ \vdots \\
&+ \sum_{i_1>i_2>\dots>i_{m-1}=1}^m \|K_{V_{i_1} V_{i_2} \dots V_{i_{m-1}}}\| \dim G\left[\bigcup_{i \in \{i_1, i_2, \dots, i_m\}} K_{V_i} \right]
\end{aligned} \tag{4.8}$$

This completes the proof of theorem 4.1.

This theorem allows us to show in the corollary below that the dimension of an arbitrary union of complete graphs of equal order is simply the dimension of the complete graphs.

Corollary 4.1. *Let $K_{V_1}, K_{V_2}, \dots, K_{V_m}$ be the maximal cliques in the minimum clique*

cover of G . If $|V_1| = |V_2| = \dots, |V_m| = N$ then, $\dim G = \dim K_N = N - 1$. Furthermore, every vertex in the graph has a regular dimension of $N - 1$.

Proof: We will do an inductive proof. Let $K_L = \bigcap_{i=1}^m K_{V_i}$. The case of $m = 2$ follows immediately from the formula for $\dim G$ in lemma 4.1.

$$\begin{aligned} (|G| - |K_L|)\dim G &= \|K_{V_1}\|\dim K_N + \|K_{V_2}\|\dim K_N \\ &= \dim K_N (\|K_{V_1}\| + \|K_{V_2}\|) = \dim K_N (|G| - |K_L|) \\ \dim G &= \dim K_N = N - 1 \end{aligned}$$

Assume the case is true for m maximal cliques, all of the same order N , so that $\dim G = \dim K_N$. Then, since the dimension of the graph with $m + 1$ maximal cliques is expressed as the sum of dimensions of graphs each with a clique cover number of m or less, each dimension in the sum will be the same ($\dim K_N$) by the inductive assumption. It then follows simply that

$$\begin{aligned} (|G| - |K_L|)\dim G &= \dim K_N \left(\sum_{i=1}^m \|K_{V_i}\| + \sum_{i>j=1}^m \|K_{V_i V_j}\| + \dots \right. \\ &\quad \left. + \sum_{i_1>i_2>\dots>i_{m-1}=1}^m \|K_{V_{i_1} V_{i_2} \dots V_{i_{m-1}}}\| \right) \\ &= \dim K_N (|G| - |K_L|) \\ \dim G &= \dim K_N \end{aligned}$$

To prove that every vertex in the graph has regular dimension $N - 1$, we note that the sphere at any vertex in the graph is the union of complete graphs of uniform order $N - 1$. Therefore the sphere at each vertex has dimension $N - 2$, giving the result that the dimension at each vertex is $N - 1$.

Example 4.1. The complete k -partite graph K_{n_1, n_2, \dots, n_k} has dimension $k - 1$.

For the corollary below let us restate the definition of the **expansion** of a vertex in a graph given in [2]. For $v \in V(G)$ the expansion at v of G is another graph G' with one additional vertex that is connected to v and all the neighbors of v in G .

Corollary 4.2. *Let G be a complete k -partite graph $G = K_{n_1, n_2, \dots, n_k}$, if every vertex in any one of the partitions is expanded p times (so that each vertex in that partition turns into a complete graph K_p) the dimension of the resulting graph is $k - 1 + p$.*

Proof: Note that before the expansion, the complete k -partite graph has maximal cliques of uniform order K_k . After we expand each vertex, say, in the i^{th} partition into a complete graph K_p , then the resulting graph is no longer complete k -partite because the nodes in the i^{th} partition belonging to the same expanded K_p are connected to each other. However, the expansion is done in such a way that the maximal cliques of the expanded graph are all K_{k+p} . Therefore the resulting graph has dimension $k + p - 1$.

5. BOUNDS ON THE KNILL DIMENSION

In this section we derive some bounds on the Knill dimension of a graph. The bounds are given in terms of the maximum clique number $\omega(G)$ and the minimum clique number $\gamma(G)$ of the graph defined in section 2.

Corollary 5.1. $\gamma(G) - 1 \leq \dim G \leq \omega(G) - 1$.

Proof: Both bounds follow from theorem 4.1 and corollary 4.1. Let $\omega(G) = k$ and $\gamma(G) = \ell$, the lower bound comes from turning all maximal cliques in G into K_ℓ by removing edges without changing the order of the graph. This results in a graph with dimension $\ell - 1$. The upper bound comes from turning all maximal cliques in G into complete graphs K_k by adding edges, which gives $k - 1$ for the the dimension of the graph.

Corollary 5.2. *If G is a graph with maximum clique number $\omega(G) = k$, then*

$$\frac{k(k-1)}{|G|} \leq \dim G \leq k-1. \quad (5.1)$$

Proof: The lower bound occurs for the edge-minimal graph with maximum clique number k . This is a graph with a single k -clique and every other vertex isolated, i.e., G is the disjoint union of K_k and $|G| - k$ isolated vertices. Using 3.1, $\dim G = k/|G| \dim K_k = k(k-1)/|G|$. The upper bound occurs for edge maximal graphs with clique number k , which is when all maximal cliques in G have order $k = \omega(G)$. By corollary 4.1 the dimension of G is $k-1$.

For connected graphs the lower bound in (5.1) can be improved.

Corollary 5.3. *Let G be a connected graph with maximum clique number $\omega(G) = k$, then*

$$1 + \frac{k^2(k-1)(k-2)}{|G|(k(k-2) + |G|)} \leq \dim G \leq k-1. \quad (5.2)$$

Proof: Let the edge set of the maximal clique of order k be U , so that K_U is the highest order clique in the clique cover. The lower bound comes from a graph with a single maximal clique of order k and all other maximal cliques of order 2; i.e., the union of a tree with K_U . For such a graph, every vertex that is not in K_U will be one dimensional. The vertices $v \in K_U$ will have unit spheres that are the disjoint union of a complete graph of order $k-1$ and $(\deg(v) - (k-1))$ isolated vertices. Therefore these unit spheres will have dimension $(k-2)(k-1)/\deg(v)$. It follows,

$$\begin{aligned} |G|\dim G &= \sum_{v \in K_U} \dim_G(v) + \sum_{v \notin K_U} \dim_G(v) = \sum_{v \in K_U} \left(1 + \dim S_G(v)\right) + \sum_{v \notin K_U} 1 \\ &= k + \left(\sum_{v \in K_U} \frac{(k-2)(k-1)}{\deg(v)} \right) + |G| - k \\ \dim G &= 1 + \frac{(k-2)(k-1)}{|G|} \sum_{v \in K_U} \frac{1}{\deg(v)} \end{aligned} \quad (5.3)$$

The sum $\sum_{v \in K_U} 1/\deg(v)$ can be minimized subject to the constraint $\sum_{v \in K_U} \deg(v) \leq k \left((k-1) + (|G| - k)/k \right)$. The upper bound of the constraint comes from having every vertex not in K_U be connected to a vertex in K_U to maximize the degree of the vertices in K_U . Since $1/x$ is a convex function, the inverse of the mean is always less than or equal to the mean of the inverses, and the minimum value for $\sum_{v \in K_U} 1/\deg(v)$ is achieved when every term in the sum is equal to the inverse of the average degree. In that case the degree of every vertex in K_U is $(k-1) + (|G| - k)/k$. This results in a lower bound of

$$\dim G = 1 + \frac{k^2(k-1)(k-2)}{|G|(k(k-2) + |G|)}.$$

The graphs that saturate the lower bound are graphs that look like the star graph, but with the central vertex replaced by a complete graph K_k , and where the leaves are distributed as equitably as possible among the vertices in the complete graph. See Fig. 4

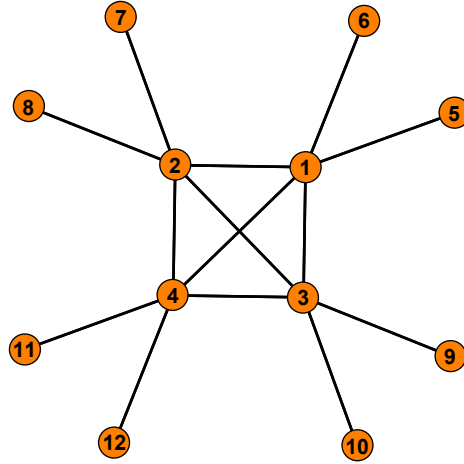


FIG. 4: A graph that saturates the lower bound of the dimension formula in (5.2) for $|G| = 12$ and $k = 4$.

6. FUTURE WORK

It will be interesting to explore the converse of corollary 4.1, i.e., to see whether or not graphs with regular integral dimension d at every vertex must necessarily be arbitrary unions of complete graphs of order $d + 1$. Our preliminary explorations indicate this to be the case. We conjecture that if a graph has uniform integer dimension d at every vertex, then all maximal cliques in the graph are K_{d+1} .

It is also desirable to find a relationship between the dimension of a graph and its complement, as highly connected graphs whose dimension is difficult to compute have complements that are sparsely connected and whose dimensions are easy to compute. The relationship between the dimension of a graph and its complement is not a straight forward one-to-one map as the scatter plot in Fig. 5 shows, however the evident structure and pattern in the scatter plot indicates that perhaps a somewhat simple relationship exists.

7. SUMMARY

The main results of this paper are the following:

1. The dimension of the join of two graphs is the sum of their dimensions plus one.

$$\dim (G_1 + G_2) = \dim G_1 + \dim G_2 + 1.$$

2. The Knill dimension of a graph has the formula given in theorem 4.1 in terms of the maximal cliques in the minimum clique cover of the graph.
3. Graphs with clique cover of uniform order, i.e, where all maximal cliques are of the same order N , have dimension $N - 1$. In addition, such a graph has regular vertex dimensions, i.e., every vertex in such a graph has dimension of $N - 1$.

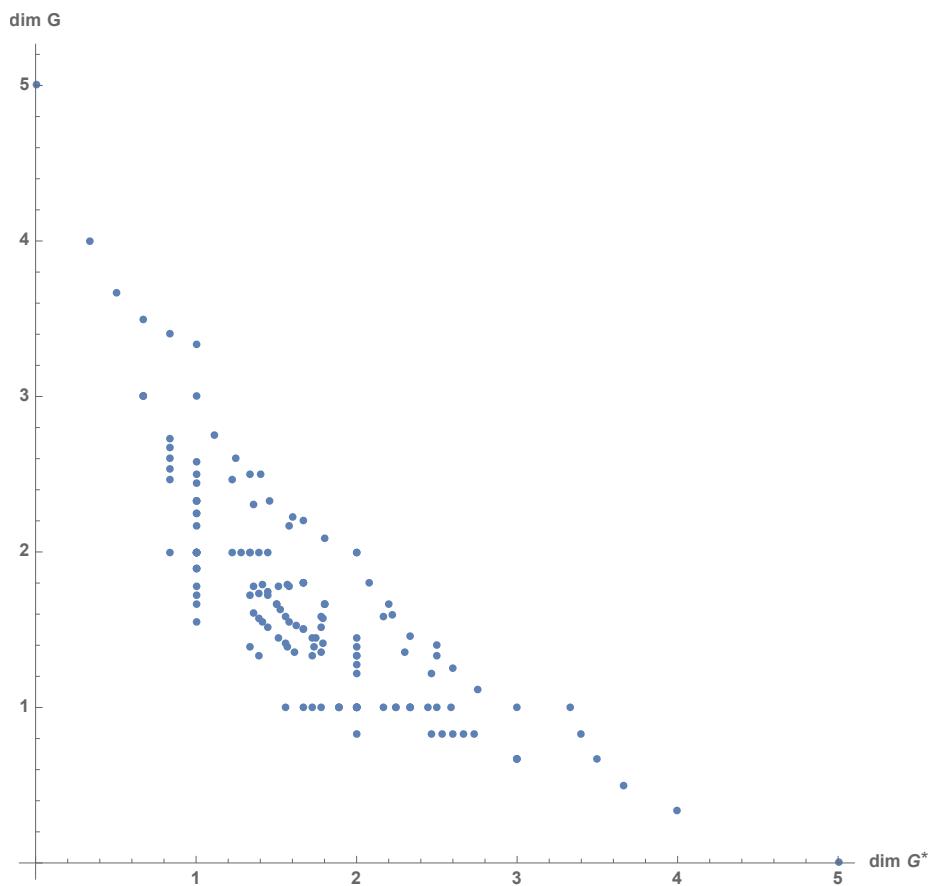


FIG. 5: Scatter plot of the dimension of a graph vs the dimension of its complement. The plot includes all non-isomorphic graphs of order 6.

4. The dimension of a graph is bounded between $k(k-1)/|G|$ and $k-1$ where $k = \omega(G)$ is the maximum clique number. If the graph is connected, the lower bound can be improved to (5.2),

$$1 + \frac{k^2(k-1)(k-2)}{|G|(k(k-2) + |G|)} \leq \dim G \leq k-1.$$

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References

- [1] Bollobas, B., “Modern Graph Theory (Graduate Texts in Mathematics),” Springer, 2002
- [2] Diestel, R., “Graph Theory (Graduate Texts in Mathematics),” 5th Edition, Springer, 2010
- [3] Knill, O. 2010, “A discrete Gauss-Bonnet type theorem,” arXiv:1009.2292
- [4] Knill, O. 2011, “On the Dimension and Euler characteristic of random graphs,” arXiv:1112.5749
- [5] Knill, O. 2015, “The Kuenneth formula for graphs,” [arXiv:1505.07518 [math.CO]].