Some Potential Means for Venture Valuation

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In some modern venture valuation approaches, option pricing theory plays an important role. The aim of this paper is to present some tools and viewpoints which might be helpful for future investigations along this line. We model the value-dynamics X_t of an imbedded underlying X as a non-lognormally-distributed generalization of the geometric Brownian motion. In detail, X_t is supposed to be a solution of a stochastic differential equation of the form

$$dX_t = b(X_t)\,dt + \sigma(t)\,X_t\,dW_t\ ,$$

with non-constant volatility function $\sigma(t)$ and Brownian motion W_t. For this, we discuss a certain decision problem concerning the size of the trend function b. Under some handy-to-verify but far-reaching assumptions, we investigate the (average) reduction of decision risk that can be obtained by observing the sample path of X. Furthermore, we also show some connections with the valuation of call options on X.

Introduction

It is well known that option models can be used in the framework of valuating ventures. For instance, Kogut (1991) considers joint ventures in terms of real options to expand in response to uncertain future technological and market developments. In such a context, the underlying is e.g. basically played by the equity value of one of the two partners. Along this line, further progress on joint venture options can be found in Chi and McGuire (1996), Folta (1998), and Chi (2000). Amongst other things, the latter also investigates the effects of the presence of options for the designing of the joint venture. For the use of option theory in connection with decisions to acquire additional equity in partner firms in research-intensive industries, see also Folta and Miller (2002). The article of Miller and Folta (2002)

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discusses a real call option approach to initial foothold investments made by firms entering new markets.

In a different framework, Hurry, Miller and Bowman (1992) work out the existence of an implicit or "shadow" option on new technology in Japanese venture capital investments in high-technology U.S. enterprises. Accordingly, the underlying is played by the value of a newly developed technology.

Keeley, Punjabi and Turki (1996) use multi-stage option pricing techniques in order to describe the value of early-stage ventures, which usually have high risk levels and involve sequential investment decisions. There, the role of the underlying is played by the value of the company in its early stages.

Finally, ordinary financial options are sometimes involved as "direct" assets of the venture, of course.

Usually, options are evaluated by (i) choosing an appropriate model for the evolution of the value of the underlying, and (ii) by applying the risk-neutral pricing approach and accordingly calculating the discounted expectations (with respect to the risk-neutral measure) of the payoff. Of course, the first step (i) is also helpful for other questions about ventures.

One popular line of underlying-models are the discrete-time binomial Cox-Ross-Rubinstein approach (1979) and its continuous-time counterpart, the geometric Brownian motion X_t, which is the unique (strong) solution of the stochastic differential equation (SDE)

$$dX_t = c\, X_t\, dt + \sigma\, X_t\, dW_t \;,\; t \in [0, T],$$
$$X_0 = x > 0 ,$$

(1)

with some constants $c \in \mathbb{R}$ and $\sigma > 0$, fixed final time horizon $T > 0$, and Brownian motion W (see Samuelson (1965), Merton (1969), and Merton (1971)). In other words, X_t follows a diffusion process with linear trend (drift) function $b(y) := c\,y$ and constant volatility $\sigma > 0$. Accordingly, the corresponding risk-neutral value of a European call option on the underlying X is given by the well-known Black-Scholes formula (1973) (see also Merton (1973)).

However, as it can be seen from the first few paragraphs above, the value of the underlying in ventures-concerning options is typically subject to a non-constant-volatility dynamics. Hence, one reasonable step of generalization is to model the evolution X_t of the underlying-value by the generalized geometric Brownian motion described by the SDE

$$dX_t = c\, X_t\, dt + \sigma_t\, X_t\, dW_t \;,\; t \in [0, T],$$
$$X_0 = x > 0 ,$$

(2)

with some deterministic nonnegative volatility function $\sigma_t := \sigma(t)$. For example, σ_t might be chosen as increasing during a first time-period and decreasing afterwards. The corresponding probability law of (2) will be denoted by $P^c_{(0,x)}$; with respect to this law, X_t is log-normally distributed.

Since ventures often take place in a very "trendy" environment, it also makes sense to use underlying-value evolution models of the form

$$dX_t = b(X_t)\, dt + \sigma_t\, X_t\, dW_t \;,\; t \in [0, T],$$
$$X_0 = x > 0 .$$

(3)

In other words, X_t is a diffusion process with trend function $b(y)$ and non-constant volatility function σ_t. For instance, one could take $b(y)$ as strongly increasing in order to mimic a current boom. The corresponding probability law of (3) will be denoted by $Q_{(0,x)}$; with respect to this law, X_t is now typically *non*-lognormally distributed. As a technical side remark, notice that we have fixed X_t and consequently we identify the models (2) and (3) by their respective solution laws; for the questions addressed in this paper, this can always be achieved by working on the probability space of sample paths (strictly speaking, one would also have to use different Brownian motions).

Certainly, it makes sense to investigate questions in the vicinity of the two following topics: to select a model for the underlying-value dynamics, and to give the adequate option price formula.

The paper is organized as follows: as a preliminary, in Section I we describe a statistical procedure for deciding (testing) between the "usual-trend-model" (2) and the "unusual-trend-model" (3). In Section II we present handy-to-verify but yet far-reaching assumptions on the trend function b and the volatility function σ_t, such that all the assertions work out properly; these assumptions can be considered as a verification-toolbox for the financial engineer when designing a venture (allowing for a wide variety of different underlyings). With this in hand, as a follow-up of Section I we give some estimates upon the there-involved decision risk reduction over time. The results of Section II can be used in order to derive the call option price formula for the model (3); this will be shown in Section III.

I.   A Model Decision Procedure

For this section, let us fix the trend constant $c \in \mathbb{R}$, the trend function b, the volatility function σ_t, and the starting underlying-value $x > 0$.

Assume that you don't know whether the time-evolution of the value of the underlying X (or a closely related object) is better described by the model (3) with trend function $b(y)$ or by the model (2) with the linear trend function $\widetilde{b}(y) := c\,y$. Accordingly, suppose that you want to decide, in an optimal way, which degree of evidence γ you should attribute (according to a pregiven loss function L) to the "event" that X has trend function b. Note that the volatility function is not object of the decision.

As one (Bayesian) way to achieve this goal, you can choose a loss function $L(\theta, \gamma)$ defined on $\{0,1\} \times [0,1]$; this represents the loss/error which arises when the (unknown) parameter is of value θ and the actually taken decision is γ. Furthermore, according to your beliefs (or experiments) prior to time 0, you fix a prior (binomial) probability $p \in]0,1[$ for the event $\theta = 1$, which is associated with the general-trend-bearing law $Q_{(0,x)}$. Also, you attach the prior (binomial) probability $1 - p$ to the event $\theta = 0$, which is associated with the linear-trend-bearing $P^c_{(0,x)}$. It is assumed that the prior probability p should not depend on x.

The risk (or uncertainty), *prior* to time 0, from the optimal decision about the degree of evidence γ concerning the decision parameter θ, is defined as

$$BR_L(p) := \inf_{\gamma \in [0,1]} \{ (1 - p) L(0, \gamma) + p L(1, \gamma) \} ;$$

this can be interpreted as a minimal prior expected loss.

In order to reduce the decision risk, imagine that you plan to observe a sample path of X_t in the time interval $[0, T]$. The corresponding risk (or uncertainty), *posterior* to the observation of X, from the optimal decision about the degree of evidence γ concerning the parameter θ, is given by

$$BR_L \left(p \parallel\!\parallel Q_{(0,x)}, P^c_{(0,x)} \right) := \int_\Omega BR_L(p_{post}) \left(p \, dQ_{(0,x)} + (1 - p) \, dP^c_{(0,x)} \right), \qquad (4)$$

with posterior probabilities $\quad p_{post} := \dfrac{p \, Z^c_{0,T}}{p \, Z^c_{0,T} + (1 - p)} \quad$, where

$$Z^c_{0,T} := \frac{dQ_{(0,x)}}{dP^c_{(0,x)}} := \exp \left(\int_0^T \frac{b(X_v) - c \, X_v}{\sigma_v \, X_v} \, dW_v - \frac{1}{2} \int_0^T \frac{(b(X_v) - c \, X_v)^2}{\sigma_v^2 \, X_v^2} \, dv \right)$$

which satisfies $EP^c_{(0,x)}[Z^c_{0,T}] = 1$ by the well-known Girsanov theorem (1960). Here, $EP^c_{(0,x)}$ denotes the expectation with respect to the law $P^c_{(0,x)}$. To justify (4), in the usual way of Bayesian decision theory we use the concept of decision rules D, which in our context are functions defined on the space of all possible sample paths - restricted to the time interval $[0, T]$ - of the process X; here, $D(X|_{[0,T]}) \in [0,1]$ gives the decision (upon $\gamma \in [0,1]$) to be taken when the actually observed sample path of X in the time interval $[0, T]$ is given by $X|_{[0,T]}$. The corresponding average loss (called frequentist risk) is given by

$$FR(\theta, D) := E_\theta[L(\theta, D(X|_{[0,T]}))]$$

with $E_0 := EP^c_{(0,x)}$ and $E_1 := EQ_{(0,x)}$. The integrated risk

$$IR(p, D) := E^p[FR(\theta, D)] := p \, FR(1, D) + (1 - p) \, FR(0, D)$$

describes the frequentist risk averaged over the values of θ according to their prior distribution. Any admissible decision rule D^* which minimizes this integrated risk $IR(p, D)$ is called a Bayesian decision rule; the corresponding minimal value $IR(p, D^*)$ (which is called Bayes risk) is equal to the term $BR_L \left(p \parallel\!\parallel Q_{(0,x)}, P^c_{(0,x)} \right)$ given in (4) above. Indeed, for any prior probability p and any decision rule D one can calculate

$$IR(p, D) = p\, EQ_{(0,x)}[L(1, D(X|_{[0,T]}))] + (1 - p)\, EP^c_{(0,x)}[L(0, D(X|_{[0,T]}))]$$

$$= EP^c_{(0,x)}[\, p\, Z^c_{0,T}\, L(1, D(X|_{[0,T]})) + (1 - p)\, L(0, D(X|_{[0,T]}))\,]$$

$$= EP^c_{(0,x)}[\{\frac{p\, Z^c_{0,T}}{p\, Z^c_{0,T} + 1 - p}\, L(1, D(X|_{[0,T]})) + \frac{1 - p}{p\, Z^c_{0,T} + 1 - p}\, L(0, D(X|_{[0,T]}))\}$$

$$\times (p\, Z^c_{0,T} + 1 - p)\,]$$

$$= EP^c_{(0,x)}[\{\, p_{post}\, L(1, D(X|_{[0,T]})) + (1 - p_{post})\, L(0, D(X|_{[0,T]}))\}\, (p\, Z^c_{0,T} + 1 - p)\,]$$

$$\geq EP^c_{(0,x)}[\, BR_L(p_{post})\, \Big(p\, Z^c_{0,T} + 1 - p\Big)] = \int_{\Omega} BR_L(p_{post})\, \Big(p\, dQ_{(0,x)} + (1 - p)\, dP^c_{(0,x)}\Big), \quad (5)$$

where the inequality above becomes an equality if one chooses D to be the Bayesian decision rule

$$D^*(X|_{[0,T]}) := \arg\min_{\gamma \in [0,1]}\Big\{(1 - p_{post})\, L(0, \gamma) + p_{post}\, L(1, \gamma)\Big\}.$$

Depending on the outcome of the sample path (which governs p_{post}), one decides accordingly.

Having just finished the verification of (4), let us proceed with the introduction of the quantity

$$\Delta BR_L\Big(p, Q_{(0,x)}, P^c_{(0,x)}\Big) := BR_L(p) - BR_L\Big(p \parallel\!\!\mid Q_{(0,x)}, P^c_{(0,x)}\Big) \geq 0 \ ,$$

which represents the reduction of the decision risk about the degree of evidence γ concerning the parameter θ, that can be attained by observing the sample path of X in the time interval $[0, T]$.

One reasonable question in this context, which can be linked with the derivation of option pricing results (see Section III below) is the following: how much is the (average) reduction of the decision risk which is contained in the above-mentioned Bayesian decision problem? Clearly, the answer to this question depends essentially on the choice of the loss function L; some corresponding results for two different kind of loss functions will be presented in the next Section II.

Of course, the method described in the current Section I works analogously for subperiods $[t_i, t_{i+1}]$ instead of the overall period $[0, T]$, with $0 = t_0 < t_1 < ... < t_n = T$. This can be used to build up an updating (sequential) decision procedure, by choosing the posterior probability obtained at the end of the period $[t_i, t_{i+1}]$ as the prior probability at the beginning of the subsequent period $[t_{i+1}, t_{i+2}]$, and so on. Such a multi-stage approach fits e.g. very well to the above-mentioned framework used in Keeley, Punjabi and Turki (1996).

Also, in the case that one is only interested in deciding between the two models (2) and (3), one can for instance stop the sample-path-observation at the first time $\tau \leq T$ (if it exists) at which the "uncertainty" (in the general sense of DeGroot (1962)) $BR_L(p_{post}(\tau))$ becomes less or equal than a pregiven threshold ε, where $p_{post}(\tau)$ is defined in the same way as p_{post}, with the only difference that T is replaced by τ.

Let us finally comment that, in principal, one can run everything in this Section I analogously for discrete-time (binomial) approximations of the two models (2) and (3).

II. Decision Risk Reductions

In order to study questions about the size of the (average) reduction of the decision risk, we use the following handy-to-verify but yet far-reaching assumptions on the trend function b and the volatility function σ_t (see Stummer (2001a)):

Assumption II.1 *The volatility function σ_t satisfies the two conditions*

$$\int_0^T \sigma_v^2 \, dv < \infty \tag{6}$$

and

$$\lim_{\Delta t \downarrow 0} \sup_{u \in [0,T]} \int_u^{\min\{(u+\Delta t),T\}} \frac{1}{\sigma_v^2 \sqrt{\int_u^v \sigma_{v_1}^2 \, dv_1}} \, dv = 0 . \tag{7}$$

The trend function b satisfies the condition

$$\sup_{a \in \mathbb{R}} \int_{a-1}^{a+1} \frac{\{b(e^\varsigma)\}^2}{e^{2\varsigma}} \, d\varsigma < \infty . \tag{8}$$

As soon as e.g. the venture design process has indicated an underlying-value dynamics with a certain possible pair of trend and volatility function, one can try to verify the three conditions in Assumption II.1 in order to get automatically the decision risk reduction and option pricing results below.

In Stummer (2001a), it is shown that Assumption II.1 guarantees the existence of a (weak) solution $X_t > 0$ of the SDE (3). In that article, one can also find some examples which demonstrate in the extreme the wide range of this framework; for instance, (i) increasing/decreasing volatility functions of the form $\sigma_t := \sigma \, t^{\beta_1}$ with constant $\sigma > 0$ and some positive/negative powers β_1, and (ii) high-boost-imitating trends of the form $b(y) := |y - v|^{\beta_2}$ for some "target" $v > 0$ and some power $\beta_2 < 0$, are even covered. By the way, notice that the geometric Brownian motion model (1) is trivially covered by Assumption II.1.

Let us now present some long-/short-time estimates upon the decision risk reduction, for two different kinds of loss functions. We first illuminate the following:

Context 1. Consider the loss function $L_0(\theta, \gamma) := \gamma - (2\gamma - 1) \mathbb{1}_{\{1\}}(\theta)$, defined on $\{0,1\} \times [0,1]$, where $\mathbb{1}_A(\cdot)$ denotes the indicator function on a set A. This corresponds to the Bayesian *testing* problem $H_0 : Q_{(0,x)}$ against the alternative $H_1 : P^c_{(0,x)}$. In fact, because of the specific form of L_0 one gets easily the formulae $BR_{L_0}(p) = \min\{p, 1-p\}$ and

$$
D^*(X|_{[0,T]}) = \begin{cases} 0, & \text{if } p_{post} < \frac{1}{2}, \\ 1, & \text{if } p_{post} > \frac{1}{2}, \\ \text{any number in } [0,1], & \text{if } p_{post} = \frac{1}{2}; \end{cases}
$$

consequently, one basically ends up with choosing between the extremal evidence degrees $\gamma = 0$ or $\gamma = 1$ (because the case $p_{post} = \frac{1}{2}$ is rather rare already for computer-numerical reasons). For this situation, one can, for instance, investigate the decision risk reduction $\Delta BR_{L_0}\left(p, Q_{(0,x)}, P^c_{(0,x)}\right)$ averaged (with some weights) over all possible choices of prior probabilities p; this is useful in situations where one does not want to stick to a single p_0 (e.g. because the historic data say so). In detail, one obtains:

Theorem II.2 *If the Assumption II.1 is satisfied, then the following assertions hold:*
(a) for all $\alpha \in \mathbb{R}$:

$$
\lim_{\Delta t \downarrow 0} \sup_{x > 0} \int_0^1 \Delta BR_{L_0}\left(p, Q^{\Delta t}_{(0,x)}, P^{c,\Delta t}_{(0,x)}\right) \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} \, dp = 0, \tag{9}
$$

where $Q^{\Delta t}_{(0,x)}$ resp. $P^{c,\Delta t}_{(0,x)}$ denotes the restriction of $Q_{(0,x)}$ resp. $P^c_{(0,x)}$ to the time interval $[0, \Delta t]$ (i.e. the underlying-value-evolution process X starts at time zero and is observed until the time Δt).

(b) according to the size of α, for any starting underlying-value $x > 0$ the time evolution (with respect to $\Delta t \in [0, T]$) of the weighted-average decision risk reduction

$$
\int_0^1 \Delta BR_{L_0}\left(p, Q^{\Delta t}_{(0,x)}, P^{c,\Delta t}_{(0,x)}\right) \frac{(1-p)^{\alpha-2}}{p^{\alpha+1}} \, dp
$$

can be estimated from above by the following function $h_1(\Delta t) :=$

$$
\begin{cases} c_1 + c_2 \, \Delta t, & \text{if } \alpha = 0, \\ \dfrac{1}{\alpha(1-\alpha)} \left\{ 1 - \exp\left(c_3(\alpha) + c_4(\alpha) \, \Delta t\right) \right\}, & \text{if } \alpha \in \left]-\infty, 0\right[\cup \left]1, \infty\right[, \\ \dfrac{1}{\alpha(1-\alpha)} \left\{ 1 - \exp\left(-c_5(\alpha) - c_6(\alpha) \, \Delta t\right) \right\}, & \text{if } \alpha \in \left]0, 1\right[, \\ \left\{c_7 + c_8 \, \Delta t\right\} \exp\left(c_9 + c_{10} \, \Delta t\right), & \text{if } \alpha = 1, \end{cases} \tag{10}
$$

with some strictly positive constants c_1, c_2, $c_3(\alpha)$, $c_4(\alpha)$, $c_5(\alpha)$, $c_6(\alpha)$, c_7, c_8, c_9, c_{10};
these constants depend on the (fixed) trend function b, the (fixed) volatility function σ_t, the
(fixed) trend constant c and, as far as indicated, also on the parameter α. All these
constants are independent of the starting underlying-value x and the evolution time Δt.

The part (a) of Theorem II.2 describes the behaviour of the weighted-average decision risk reduction when one observes X only in the time interval $[0, \Delta t]$ (rather than $[0, T]$), where the time horizon Δt tends to zero. In contrast, the part (b) estimates the time-evolution of the weighted-average decision risk reduction for *any* time horizon $\Delta t \in [0, T]$. The part (a) can be used in order to derive the corresponding option price formula (see Section III).

Differently to Context 1, let us now deal with another kind of loss function:

Context 2. Consider $L_{\alpha,\xi}(\theta, \gamma) := \dfrac{\alpha^{\theta-1} \, \gamma^{\alpha-\theta}}{\xi^\alpha \, (1 - \xi)^{1-\alpha} \, (1 - \alpha)^\theta \, (1 - \gamma)^{\alpha-\theta}}$ which is defined
on $\{0,1\} \times [0,1]$, with parameters $\alpha \in \,]0,1[$ and $\xi \in \,]0,1[$. The corresponding prior risk can be computed in a straightforward manner as

$$
BR_{L_{\alpha,\xi}}(p) = \frac{1}{\xi^\alpha \, (1 - \xi)^{1-\alpha}} \min_{\gamma \in [0,1]} \left\{ \frac{1 - p}{\alpha} \left(\frac{1}{\gamma} - 1 \right)^{-\alpha} + \frac{p}{1 - \alpha} \left(\frac{1}{\gamma} - 1 \right)^{1-\alpha} \right\}
$$

$$
= \frac{1}{\xi^\alpha \, (1 - \xi)^{1-\alpha}} \left\{ \frac{1 - p}{\alpha} \left(\frac{1}{p} - 1 \right)^{-\alpha} + \frac{p}{1 - \alpha} \left(\frac{1}{p} - 1 \right)^{1-\alpha} \right\}
$$

$$
= \frac{1}{\alpha \, (1 - \alpha)} \left(\frac{p \, (1 - \xi)}{(1 - p) \, \xi} \right)^\alpha \frac{1 - p}{1 - \xi} \; . \tag{11}
$$

The corresponding Bayesian decision rule is given by

$$
D^*(X|_{[0,T]}) = p_{post} = \frac{p \, Z_{0,T}^c}{p \, Z_{0,T}^c + 1 - p} \; .
$$

In other words, one chooses the sample-path-dependent posterior probability as the degree of evidence for the validity of the model (3). For this Context 2, one gets the following estimates on the decision risk reduction which is associated with the Bayesian decision problem with loss function $L_{\alpha,\xi}$:

Theorem II.3 *If the Assumption II.1 is satisfied, then the following assertions hold :*
(a) for all $\alpha \in \,]0,1[$ and all prior probabilities $p \in \,]0,1[$:

$$
\lim_{\Delta t \downarrow 0} \sup_{x > 0} \Delta BR_{L_{\alpha, p}} \left(p, Q_{(0,x)}^{\Delta t}, P_{(0,x)}^{c,\Delta t} \right) = 0 \; .
$$

(b) according to the size of α, for any starting underlying-value $x > 0$, and any prior probability $p \in]0,1[$, the time evolution (with respect to $\Delta t \in [0,T]$) of the decision risk reduction

$$\Delta BR_{L_{\alpha,p}}\left(p, Q_{(0,x)}^{\Delta t}, P_{(0,x)}^{c,\Delta t}\right)$$

can be estimated from above by the function $h_1(\Delta t)$ given in (10) of Theorem II.2. In particular, these estimates do also not depend on the prior probability p.

Analogously to the remark after Theorem II.2, the parts (a), respectively (b), of Theorem II.3 describe the short-time, respectively long-time, behaviour of the corresponding decision risk reduction. Again, the part (a) can be used in order to derive the corresponding option price formula (see Section III). As a technical side remark, $\Delta BR_{L_{\alpha,p}}$ has to be read as

$$\lim_{\xi \to p} \Delta BR_{L_{\alpha,\xi}}.$$

For the sake of brevity, the proofs of the two Theorems II.2 and II.3 will be omitted here. They will appear elsewhere, and follow the lines of the proofs for the special case $c = 0$, which has been treated in Stummer (2001a). In principal, one can (amongst other things) make use of important information characterization results for general measures given by Österreicher and Vajda (1993).

III. Option Pricing

Finally, as an application of the above results, let us now provide the corresponding valuation theorem of European call options on underlyings whose value-evolution process X_t is the non-lognormally distributed generalization (3) of the geometric Brownian motion. Additionally, the underlying X is supposed to continuously yield dividends of the amount $\delta_t X_t \, dt$ between time t and $t + dt$, where the dividend yield δ_t is a deterministic, continuous, non-negative function of t. Furthermore, we assume the existence of a bond or bank account B, whose value-evolution is given by $B_t = \exp(\int_0^t r_v \, dv)$, where the deterministic short rate process r_t is nonnegative and continuous in t.

As usual, we also employ the standard assumptions that the lending (interest) rate is equal to the borrowing (interest) rate, that there are no transaction costs and no taxes, and that trading takes place continuously.

Theorem III.1 *Suppose that the Assumption II.1 holds. Then, the unique arbitrage-based price V_t at time t of a European call option on the underlying X with strike price $K > 0$ and expiration date T is given by*

$$V_t = X_t \exp(-\int_t^T \delta_v \, dv) \, F_N(d_1) - K \exp(-\int_t^T r_v \, dv) \, F_N(d_2), \qquad (12)$$

with

$$d_1 := \frac{\log\left(\frac{X_t}{K}\right) + \int_t^T (r_v - \delta_v + \frac{\sigma_v^2}{2})\,dv}{\sqrt{\int_t^T \sigma_v^2\,dv}} \quad \text{and} \quad d_2 := d_1 - \sqrt{\int_t^T \sigma_v^2\,dv} \ .$$

Here, $F_N(y)$ denotes the distribution function, evaluated at y, of the standard normal distribution.

Remarks:

(i) The original Black-Scholes theorem (1973) can be derived as a special case of Theorem III.1, by taking for the underlying-value evolution X_t the linear trend function $b(y) = cy$ and the constant volatility function $\sigma_t \equiv \sigma > 0$ of the geometric Brownian motion SDE (1), together with constant short rate $r_t \equiv r \geq 0$ and zero dividend yield $\delta_t \equiv 0$. (The non-stochastic interest-rate version of) Merton's theorem (1973) deals with the same SDE set-up (1), but with non-constant short rates r_t and constant dividend yield $\delta_t \equiv \delta \geq 0$; Rubinstein (1976) uses non-constant dividend yields δ_t. Those cases are also covered by Theorem III.1.

(ii) In the theory of "real options" (which are typical building blocks for "venture options"), one uses sometimes the Black-Scholes or Merton's formula, although one knows that the underlying quantity can only be approximated by a geometric Brownian motion; see e.g. Kemna (1993) and Carr (1995). As a means to support such an action plan, the non-stochastic Assumption II.1 involved in Theorem III.1 delivers a handy-to-verify, non-stochastic toolbox for obtaining a variety of non-lognormally distributed underlyings X, such that one can still valuate the corresponding call options with the Black-Scholes formula or Merton's formula, or generalized versions thereof.

(iii) Because of the specific form of (12), the corresponding compound options on the underlying option V can be valuated according to the standard theory (e.g. with a Geske-type formula). This can be used, for instance, for early-stage ventures valuation where compound options play an important role (see Keeley, Punjabi and Turki (1996)).

In order to indicate the connection with the investigations of the Sections I and II, let us shortly give the main essence of the proof of Theorem III.1 for the special case $\sigma_t \equiv \sigma > 0$, $r_t \equiv 0$ and $\delta_t \equiv 0$. First of all, the case $\alpha = 1$ of the Theorems II.2(b) resp. II.3(b) holds in particular for the special trend constant $c = 0$. This can be related to a corresponding short-term behaviour result upon the relative entropy $H(Q_{(0,x)}^{\Delta t} \| P_{(0,x)}^{0,\Delta t})$. Consequently, the Girsanov theorem can be applied "in both directions" in order to show that $P_{(0,x)}^0$ is the unique equivalent martingale measure for $(X_t, Q_{(0,x)})$. Then the assertion of Theorem III.1 follows from arbitrage theory and integration. For further details, the reader is referred to the author's article (2001b).

IV.   Summary

Some potential tools and viewpoints in the connection with the use of option pricing methods for venture valuation are given. Especially, we deal with underlyings whose value evolve in time with non-linear trend functions and non-constant volatility functions, which are reasonable models for venture components. This contrasts with some standard methods which employ the geometric Brownian motion model with linear trend function and constant volatility function. We describe a decision procedure concerning the size of the trend function, when observing a sample path of the underlying-value evolution. For this procedure, the involved decision risk reductions are presented. As a crucial application, the corresponding, comfortably computable European call option price formula is derived, under handy-to-verify but far-reaching assumptions on the trend function and the volatility function.

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