Price Bubbles of New-Technology IPOs

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Asset pricing models with atomistic agents typically relax assumptions concerning rationality and/or homogenous information in order to track endogenous bubbles. In this model, identically informed rational agents hold a Perceived Law of Motion (PLM) for a single new technology asset at IPO, yet they differ with respect to risk aversion. By mapping risk preferences to strategies, we use marginal supply and demand functions to solve for the PLM if REE holds. By relaxing the assumption of complete knowledge of agent's tastes and wealth, post-IPO bubbles emerge where the Actual Law of Motion is an amplification (bubble) of the price processes vs. the PLM.

Introduction

Homogeneous information, uniform tastes and agents' rationality are cornerstone assumptions for capital asset pricing, but prior art conclude that under these conditions, non-atomistic agents have no incentive to trade and the expected equilibrium price will be stable. These results do not correspond to observation of capital asset markets, as trade is evident and bubbles are revealed as they occasionally burst. Over the past decades, major stock-market bubbles were associated with significant impact of new technologies on the real economy (1929, 2000) or new trade mechanisms like portfolio insurance and program trading (1987). The question we address here is whether unobservability of preferences and wealth might result in bubbles, especially following new-technology IPO.

Divergence of real and financial assets from fundamental valuation has been documented since the early 80's (Shiller (1981), Grossman and Shiller (1981), Summers (1986)). Academics, mostly, define financial bubbles as deviations from fundamentals

that persist for periods that are too long to be explained by response to external shocks. The first researchers of the subject, Keynes (1930) and Hicks (1946), attributed such deviations to *speculation* which they consider rational. Both addressed speculation as a byproduct of differences in risk aversion where more risk-averse agents "sell" some of the risk to less risk-averse ones. The more risk-averse agents trade based on fundamental values while their counterparts adopt speculative strategies. It turns out that Keynes and Hicks see speculation as a tool to reallocate risk among trading agents. In this model, we segregate agents by investment strategies based on different measures of risk aversion, thus precisely fit their approach.

Equilibrium analysis of capital asset pricing with unobservable fundamentals started with Feldman's Ph.D. dissertation (1983), and followed in articles by Dothan and Feldman (1986), Detemple (1986, 1991), Gennotte (1986), Feldman (1989, 1992), Detemple and Murthy (1994) and Coles, Loewenstein and Suay (1995). In general, the goal in the above mentioned models is to explicitly solve for the unobservable moments endogenously. They view the price process as a noisy realization of the unobservable fundamental moments. In most cases, the realized price process is a function of the conditional unobservable moments of the production factors, production function and agents' first and second utility function derivatives. Another approach has been taken by Kurz (1994a, 1994b, 1996 and 1997) who constructed the theory of Rational Beliefs. According to Kurz, agents do not possess complete structural knowledge of the environment and its changes, especially changes in technology, tastes and economic institutions. Thus, there is a question whether such changes are random deviations around a fixed mean value function of a stationary process, or whether they reflect changes in the mean value function itself. Beliefs must satisfy Kurz's Rationality Principle, that call for consistency of beliefs with the data. Essentially, beliefs are formed by assuming agents act as econometricians who apply a learning model on realized returns and use the coefficients to plan ahead.

Rational expectations models of bubbles are attractive as they adhere to core economic theory. This is why these models were the first to appear in the late '70s - early '80s (Blanchard (1979), Flood and Garber (1980), Blanchard and Watson (1982), Tirole (1982)). Bubbles in these models evolve based on a self-fulfilling rational expectations mechanism, which supports the bubble as long as it exists, and the price crashes when the mechanism disappears. These models do not specify what endogenous conditions must be satisfied for a bubble to evolve, knowing that a stable economy have operated with the same agents and technology prior to the boom, and under what inherent terms will the bubble blast. Starting from the mid '80s equilibrium models of bubbles typically involve explicit or implicit assumptions of irrational behavior by a certain group of agents. Some models assume noise trading (e.g., Kyle (1985), Black (1986), De-long, Shleifer, Summers and Waldmann (1990), Binswanger (1999), and Levy, Levy and Solomon (2000). Others assume overreaction to news (Jegadeesh and Titman (1995), De-Bondt and Thaler (1987), Kent, Hirshleifer and Subrahmanyam (1998)) or signaling between investors who have asymmetric information (Allen and Gale (1992), (2000a), (2000b), Allen and Gorton (1993), Allen, Morris and Postlewaite (1993)). The economic meaning of such assumptions is that either fundamental information is not available to all agents (asymmetric information), and/or different agents process information in different ways (the noise-trader approach). Binswanger (1999) extends De-Long et al.'s model by

allowing dynamic changes of the fundamentals due to technological changes. On another front, the emergence of derivative securities and thereby Portfolio Insurance (PI) is claimed to contribute to the underlying assets volatility if markets are not complete (Grossman (1988a), (1988b), Grossman and Zhou, (1996)).[1] Research about PI boosted after 1987 when academia and government-committees have studied their potential contribution to the stock market crash (Brady, 1988, SEC, 1987).

In this paper we show that agents' inability to observe others' optimal trading strategies, classified to Momentum and Contrarian, might result in amplified volatility and return if the Momentum strategy dominates or a dimmed price process if the Contrarian dominates. We thus specify explicitly positive (and negative) mispricing, while observability yields the Merton (1971) rational expectations process. This model uses the notion of incomplete information at and post-IPO to establish a model of bubbles, thus should not be considered as a model of IPO.

Section I describes optimal dynamic investment rules for non-price-taking agent groups, based on Merton (1971). In Section II we define the groups, derive their marginal supply and demand functions for shares and solve the equilibrium price. In Section III we relax the complete information assumption and derive ex-post moments vs. Rational Expectations Equilibrium (REE), resulting in Momentum Dominance Equilibria (MDE) or Contrarian Dominance Equilibria (CDE). In section IV we discuss bubbles and Section V concludes.

I. The Economic Setting

Consider a reduced form of Merton (1971) economy in which we address the portfolio selection problem with incomplete information[2]. The investment opportunity set is comprised of two assets, a riskless bond yielding exogenous, fixed rate of return r and a single risky asset. The latter represents a new-technology equity share for which there is no return and variability track-record.[3] All agents have the same information thus

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[1] In order to insure a portfolio, an investor may either hold the underlying risky-asset long and buy a put option with a strike equal to the desired insurance level, or he may buy a call option with a risk-free asset. In both cases, the negative tail of the distribution is sold (for a fair premium) to the issuer of the option. The latter may bare the risk, and the underlying asset price will not be affected, or he may replicate a Portfolio Insurance (PI) strategy in the underlying securities market to transfer his risk to other investors. By doing so, such agent is executing a momentum (speculative) trading strategy vs. "the market" as long as he is atomistic. However, if the investor affects market price, the latter will rise and eventually revert to the fundamental. It turns out that by modeling momentum strategies in the underlying assets we capture the spillover of the demand for portfolio insurance to the contrarian agents, who are essentially suppliers of PI.
[2] Merton (1971) shows that the effect of a fixed planning horizon on the intertemporal portfolio-selection problem comes through the riskless capitalization factor $(1-e^{r(t-T)})/r$, applied on a constant displacement in the HARA-type utility function, termed η. This function is strictly concave and monotonically declines in t. For a long enough horizon T, this factor is marginally constant when T is far ahead and declines at faster rates as $t \to T$, reaching 0 at T. Particularly throughout the last phase, systematic effects on the stock weight in the portfolio emerge. Since we want to present different portfolio effects, we shall ignore the time-effects by assuming that T is far enough for all agents, an assumption that corresponds with IPO at $t=0$.
[3] In the Merton (1971) model there is no "hedging demand" which implies a certain degree of myopia that the present model inherits. Since the context here is for new-technology shares with idiosyncratic risk, this short-coming may be of less importance. Alternatively, as noted in Ross (1975), one may consider the above investors as institutions for which the myopia deficiency is less important than for an individual investor.

the return generating process of the stock is assumed by all to follow a discrete-time Geometric Brownian Motion, being the Perceived Law of Motion (PLM). We assume that agents do not know each other's wealth and utility function. We classify atomistic, expected utility maximizing individual agents based on the direction of their *marginal* trade, adopting the terms "Momentum" and "Contrarian" strategies, as will formally be presented below.

New information about the real capital asset's value arrives periodically through a normally distributed noise z_t. Due to the unobservability of other agents' marginal trade, we establish the equilibrium price through trade in a tatonnement clearing mechanism. Since we endogenously solve for the return generating process, we (and not the agents) can compare it with Merton's (1971) REE price process and specify positive or negative bubbles, as if an equivalent economy existed. Finally, we assume that there are no transaction costs, taxes, or other frictions.

A. Agents

Agents are classified by two generic strategies - *Momentum (M)*, whereby an increase in wealth call for increasing the amount invested in shares, and *Contrarian (C)*, according to which an increase in wealth result in reducing the amount invested in shares. Strategies and their magnitude depend on the specific HARA-type utility function parameters, as illustrated below.

The PLM is a Geometric Brownian Motion with constant $\hat{\mu}$ and $\hat{\sigma}$, $d\hat{P}_{t+\Delta t} = P_t(\hat{\mu}\Delta t + \hat{\sigma}z_t\sqrt{\Delta t})$. Under complete information and Rational Expectations (RE) the realized stock price moments must comply with the PLM. Let $S_{k,t} = N_{k,t}P_t, \forall t$ be the stock value held by individual agent k at t, where $N_{k,t}$ is the number of shares and P_t their price. $\sum_k N_{k,t} = N$ is the number of shares issued at the IPO. Define bond value held by the agent $D_{k,t} = Q_{k,t}B_t$, $\forall t$, where Q is quantity of bonds and B their price (bond are available to all agents at unlimited supply), and let $\alpha_{k,t} = S_{k,t}/W_{k,t}$ and $1 - \alpha_{k,t} = D_{k,t}/W_{k,t}$. W is the wealth invested in stock and bond. All $t = 0$ values represent IPO allocation. Thus

$$
\begin{aligned}
\hat{W}_{k,t+\Delta t} &= N_{k,t}\hat{P}_{t+\Delta t} + Q_{k,t}B_{t+\Delta t} \\
&= S_{k,t}(1 + \frac{d\hat{P}_{t+\Delta t}}{P_t}) + D_{k,t}(1 + r\Delta t) \\
&= S_{k,t}(1 + \hat{\mu}\Delta t + \hat{\sigma}z_t\sqrt{\Delta t}) + D_{k,t}(1 + r\Delta t) \\
&= W_{k,t}\left(1 + \alpha_{k,t}(\hat{\mu}\Delta t + \hat{\sigma}z_t\sqrt{\Delta t}) + (1 - \alpha_{k,t})r\Delta t\right)
\end{aligned}
\tag{1}
$$

Such a wealth-conservation process[4] is frequently presented as

$$
\hat{W}_{k,t+\Delta t} = W_{k,t}\left(1 + (r + \alpha_{k,t}(\hat{\mu} - r))\Delta t\right) + \alpha_{k,t}W_{k,t}\hat{\sigma}z_t\sqrt{\Delta t}.
\tag{2}
$$

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[4] Note that the last equality is equivalent to equation [10] in Merton's (1971) discrete time prolog to the continuous time model.

Samuelson (1969), Friend and Blume (1975) and Ross (1975) were the first to analyze such wealth accumulation processes in discrete-time while the continuous-time equivalent has been pioneered by Merton (1969, 1971 and 1973). Using a Taylor-series expansion to estimate $U(\hat{W}_{k,t+\Delta t})$ up to the second order, where U is a von-Neumann-Morgenstern utility function, and applying the expectation operator, we get[5]

$$E\left(U(\hat{W}_{k,t+\Delta t})\right) = U(W_{k,t}) + U'(W_{k,t})W_{k,t}\left(r + \alpha_{k,t}(\hat{\mu} - r)\right)\Delta t + \frac{1}{2}U''(W_{k,t})W_{k,t}^2\alpha_{k,t}^2\hat{\sigma}^2\Delta t. \quad (3)$$

The first order condition of (3) with respect to $\alpha_{K,t}$ is,

$$\frac{dE(U)}{d\alpha_{k,t}} = U'(W_{k,t})W_{k,t}(\hat{\mu} - r) + U''(W_{k,t})W_{k,t}^2\alpha_{k,t}\hat{\sigma}^2 = 0. \quad (4)$$

Solving for $\alpha_{k,t}$ we get

$$\alpha_{k,t} = -\frac{U'(W_{k,t})}{U''(W_{k,t})W_{k,t}}\frac{\hat{\mu} - r}{\hat{\sigma}^2}, \quad (5)$$

and by using the Arrow-Pratt measure of relative risk aversion, $R_{W,k,t} \equiv -\dfrac{U''(W_{k,t})W_{k,t}}{U'(W_{k,t})}$, we obtain the familiar relationship, similar in form to the models of the aforementioned authors,

$$\alpha_{k,t} = \frac{1}{R_{W,k,t}}\frac{\hat{\mu} - r}{\hat{\sigma}^2}. \quad (6)$$

Let all individuals have a hyperbolic utility function in wealth of the following HARA type

$$U(W,t) = e^{-\rho t}\frac{1-\gamma}{\gamma}\left(\frac{W}{1-\gamma} + \eta\right)^\gamma. \quad (7)$$

We use this utility function since agents with different attitudes toward risk apply different dynamic strategies that are simple to obtain from (7) and are the building blocks of this model. This function can produce relative or absolute measures of risk aversion that both can be constant, decreasing or increasing in wealth. Table 1, in Appendix 1, summarizes this function's properties. By fixing $R_{W,k,t}$ in (6) we obtain the optimal investment rule for individual agent k,

$$\alpha_{k,t+\Delta t}^*\hat{W}_{k,t+\Delta t} = N_{k,t+\Delta t}^*\hat{P}_{t+\Delta t} = \frac{\hat{\lambda}}{\delta_k}\left(N_{k,t}\hat{P}_{t+\Delta t} + Q_{k,t}B_{t+\Delta t} + \eta_k\delta_k\right), \quad (8)$$

where $\delta_k = 1 - \gamma_k$, $\hat{\lambda} = \dfrac{\hat{\mu} - r}{\hat{\sigma}^2}$.[6]

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[5] Note that neglecting changes higher than the second order when share prices follow log-normal distribution does not affect total demand for shares (specifically the demand for "hedging", elaborated in Merton (1973)).

[6] This type of utility functions embodies a displacement factor $\eta_K\delta_K$ that, if negative, represents a demand for "safety net", or insurance level to the agents' wealth. It implies replication of a call option by Constant Proportion Portfolio Insurance (CPPI) with a multiplier $m = \lambda/\delta_M > 1$ and a floor $\eta_M\delta_M < 0$. Alternatively, agents who have a positive displacement factor essentially supply portfolio insurance to the market, thus mimicking a put option. A related notion has been presented by Leland (1980), Grossman and Vila (1989), Black and Perold (1992) Grossman and Zhou (1996) and others.

II. Marginal Supply and Demand Functions for Shares

In models with atomistic investors, the stochastic price path is exogenous, observable, and unaffected by the agent's trade; thus, the equilibrium path is the stochastic process itself. However, in a partially observable economy we need an additional degree of freedom to solve the system, and that is provided by the marginal supply and demand functions for shares.

By definition, the number of shares held by agent k at $t+\Delta t$ is equal to the number of shares she held at t, plus an optimal, unknown at that stage, marginal trade over Δt, $\Delta N_{k,t+\Delta t} = N_{k,t+\Delta t} - N_{k,t}$. Henceforth lowercase k, c, and m indicate individual variables while capital letters indicate group-wise aggregate equivalents. Aggregation by group is permissible since both risk parameters η and δ are assumed identical group-wise. Note that while individual agents are atomistic, group-aggregates determine the share price endogenously. We omit the optimality symbol "*" in (8) from now until the end of this section, when we derive equilibrium, for simplicity of notation.

A. The Contrarian Strategy

Denote the aggregation of contrarian agents by subscript "C". By replacing $N_{C,t+\Delta t} = N_{C,t} + \Delta N_{C,t+\Delta t}$ in (8) and solving for $\Delta N_{C,t+\Delta t}$ we obtain

$$\Delta N_{C,t+\Delta t} = \frac{\hat{\lambda}}{P_{t+\Delta t}^d \delta_C}\left(\tilde{D}_{C,t+\Delta t} + \eta_C \delta_C\right) + N_{C,t}\left(\frac{\hat{\lambda}}{\delta_C} - 1\right), \tag{9}$$

where $\tilde{D}_{C,t+\Delta t} \equiv Q_{C,t} B_{t+\Delta t}$ and $P_{t+\Delta t}^d$ represents the marginal *demand* function for shares. Equation (9) can be rewritten as

$$P_{t+\Delta t}^d = \frac{\dfrac{\hat{\lambda}}{\delta_C}\left(\tilde{D}_{C,t+\Delta t} + \eta_C \delta_C\right)}{N_{C,t}\left(1 - \dfrac{\hat{\lambda}}{\delta_C}\right) + \Delta N_{C,t+\Delta t}}, \tag{10}$$

that will satisfy the properties of a demand function if it maintains a strictly negative slope $\dfrac{\partial P_{t+\Delta t}^d}{\partial(\Delta N_{C,t+\Delta t})} < 0$ and weakly-positive convexity $\dfrac{\partial P_{t+\Delta t}^d}{\partial(dN_{C,t+\Delta t}^2)} \geq 0$.[7] The class of agents for whom the contrarian strategy is optimal is defined in Proposition 1:

Proposition 1:

Agents who adopt a Contrarian strategy must have taste parameters $\delta_C > \hat{\lambda}\dfrac{N_{C,t}}{N_{C,t+\Delta t}}$, and $\eta_C > -\dfrac{\tilde{D}_{C,t+\Delta t}}{\delta_C}$ which imply that they have DRRA and DARA attitude

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[7] In this case, strict positive convexity will hold due to the first requirement as presented in Appendix 2.

$$\text{toward risk if} \begin{cases} -\tilde{D}_{C,t+\Delta t}/\delta_C < \eta_C < 0 \\ \delta_C > \hat{\lambda}\dfrac{N_{C,t}}{N_{C,t+\Delta t}} \end{cases}, \text{ DARA alone if } \begin{cases} \eta_C > 0 \\ \delta_C > \hat{\lambda}\dfrac{N_{C,t}}{N_{C,t+\Delta t}} \end{cases}, \text{ and CRRA if}$$

$$\begin{cases} \eta_C = 0 \\ \delta_C > \hat{\lambda}\dfrac{N_{C,t}}{N_{C,t+\Delta t}} \end{cases}.$$

Proof: See Appendix 2.

B. The Momentum Strategy

Let type *M* individual agents have a convex payoff schedule, being an investment rule whereby an increase in wealth result in an increasing exposure to shares in their portfolio, and vice versa. Following the procedure as detailed above, their marginal trade is

$$\Delta N_{M,t+\Delta t} = \frac{\hat{\lambda}}{P_{t+\Delta t}^s \delta_M}\left(\tilde{D}_{M,t+\Delta t} + \eta_M \delta_M\right) + N_{M,t}\left(\frac{\hat{\lambda}}{\delta_M} - 1\right), \tag{11}$$

where $P_{t+\Delta t}^s$ is the marginal supply function for shares, which may be phrased,

$$P_{t+\Delta t}^s = \frac{\dfrac{\hat{\lambda}}{\delta_M}\left(\tilde{D}_{M,t+\Delta t} + \eta_M \delta_M\right)}{N_{M,t}\left(1 - \dfrac{\hat{\lambda}}{\delta_M}\right) + \Delta N_{M,t+\Delta t}}. \tag{12}$$

A marginal supply function for shares must satisfy strictly positive slope $\dfrac{\partial P_{t+\Delta t}^s}{\partial(\Delta N_{M,t+\Delta t})} > 0$ and weakly positive convexity $\dfrac{\partial P_{t+\Delta t}^s}{\partial(\Delta N_{M,t}^2)} \geq 0$, although in our case strict convexity will hold. The taste parameters that satisfy a Momentum investment strategy comply with Proposition 2.

Proposition 2:
Individual agents who adopt a Momentum strategy must have a measure of risk aversion and displacement factor that satisfy $\begin{cases} \eta_M < 0 \\ 0 < \delta_M < \hat{\lambda}\dfrac{N_{M,t}}{N_{M,t+\Delta t}} \end{cases}$, *which complies with DRRA and DARA attitude toward risk. Such agents will not have CRRA or CARA attitudes toward risk.*

Proof: See Appendix 3.

A graphical illustration of the marginal supply and demand functions derived above for shares is presented in Figure 1.[8] In the following section we establish Walrasian equilibrium based on these marginal functions.
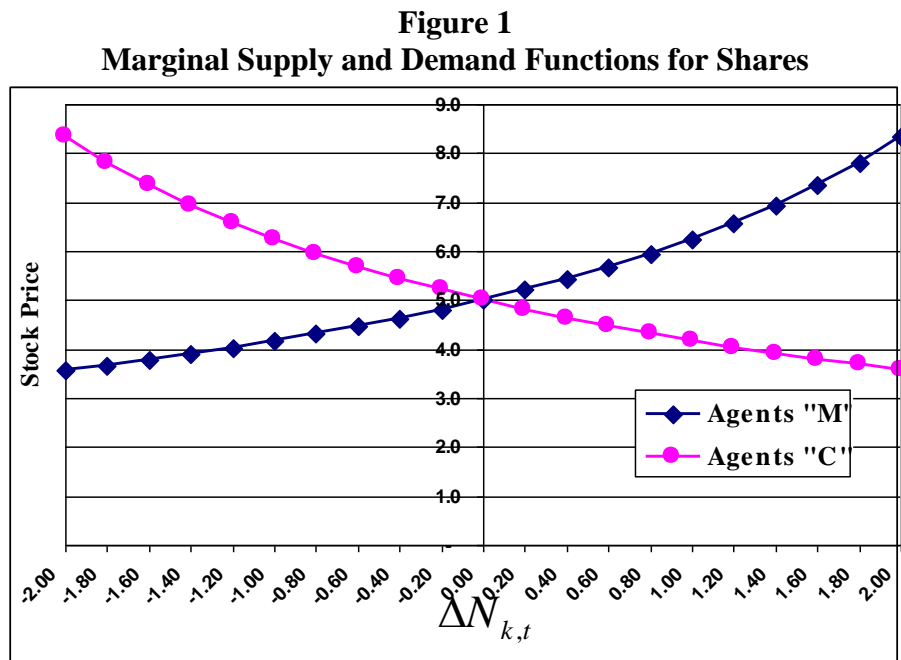
Figure 1
Marginal Supply and Demand Functions for Shares

Figure 1 describes graphically the marginal supply and demand functions for shares by agents *M* and *C*. An increase in price implies an increase in the number of shares by agent *M* ($\Delta N_{M,t} > 0$) and a decrease in the number of shares by agent *C* ($\Delta N_{C,t} < 0$). Note that for a given price increase, the horizontal gap between both functions is the Bid-Ask spread. Finally note that the "supply" and "demand" tags were attached to these functions in order to correspond to their common representation in economics, yet their names should really been reversed. The values we used to draw these functions are:

$$\hat{\lambda} = 0.6, \eta_M = -166.7, \delta_M = 0.4, \eta_C = 20.8, \delta_C = 1.2, N_M = N_C = 10, D_M = 50, D_C = 25$$

C. Equilibrium

Though taste parameters remain fixed, aggregate marginal supply/demand change periodically due to changes in wealth allocation among agent groups. Individuals submit matching vectors of quantities and share prices, being their marginal supply/demand function, to a clearing agency assumed to operate in a tâtonnement procedure. The tâtonnement aggregates individual supply and demand functions and by clearing excess marginal demand set a new equilibrium price each period.[9] We show that the resulting price path satisfies Pareto-optimality, yet rational-expectations equilibrium hold if the

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[8] These marginal supply and demand functions for shares also provide a basis for trade under a certain group of REE equilibria, among them the Merton (1971) model, yet, this issue is beyond the scope of this paper.

[9] Note that since the marginal functions are convex and defined on the same plane for all agents, aggregation is allowed across individual tastes.

PLM equal the ALM, being equivalent to Merton's (1971) price process. The Walrasian equilibrium conditions are,

$$\begin{cases} P_{t+\Delta t}^d = P_{t+\Delta t}^s \\ \Delta N_{C,t+\Delta t} = -\Delta N_{M,t+\Delta t} \end{cases}.$$ (13)

By using $N_{C,t} + N_{M,t} = N, \forall t$ in (9) and (11) and using the wealth-conserving budget constraint (1), we solve for the equilibrium price of shares $P_{t+\Delta t}^*$,

$$\begin{aligned} P_{t+\Delta t}^* &= \frac{\hat{\lambda}}{N} \left(\frac{\tilde{S}_{M,t+\Delta t}^*}{\delta_M} + \frac{\tilde{S}_{C,t+\Delta t}^*}{\delta_C} + \frac{\tilde{D}_{M,t+\Delta t}^*}{\delta_M} + \frac{\tilde{D}_{C,t+\Delta t}^*}{\delta_C} + \eta_M + \eta_C \right) \\ &= \frac{\hat{\lambda}}{N} \left(\frac{\tilde{W}_{M,t+\Delta t}^*}{\delta_M} + \frac{\tilde{W}_{C,t+\Delta t}^*}{\delta_C} + \eta_M + \eta_C \right) \end{aligned}.$$ (14)

Asterisks represent Pareto-optimal equilibrium values. Equation (14) translates to the continuous time equivalent when $\Delta t \to 0$,

$$P_t^* = \frac{\hat{\lambda}}{N} \left(\frac{W_{M,t}^*}{\delta_M} + \frac{W_{C,t}^*}{\delta_C} + \eta_M + \eta_C \right).$$ (15)

This equilibrium price is consistent with previous results, being a weighted-average value of wealth managed by each group, weighted by the divergence of agents' measure of risk aversion δ_K from $\hat{\lambda}$. Here however the displacement effect of $\eta_M + \eta_C$ plays an important role as it distinguishes strategies by marginal trade[10]. Comparing to popular utility functions, if $\eta_M = \eta_C = 0$ then, by Table 1, (15) reduces to the CRRA case. It turns out that this solution for the HARA-type utility function is a generalization of Ross's (1975) and Friend and Blume's (1975) results for multiple agents based on CRRA utility functions. Further, if we assume that all agents have identical risk parameters, $(\delta \equiv \delta_M = \delta_C; \eta \equiv \eta_M = \eta_C)$, (15) reduces to Merton's (1971) optimal investment rule with a single agent (ibid. equation [49]). The above-mentioned authors derived their results under the assumptions of exogenous, visible price process with price taking investors. (15) is similar in form to these results but different in meaning. On one hand it shows that the equilibrium share price process satisfy a Pareto-optimum asset allocation each period, regardless of the existence of REE. On the other hand, price path (15) is determined by the risk-aversion-weighted wealth of all agents M and C who hold the asset.

III. Temporary deviations from REE - Bubbles

We argue that if the assumption that all agents know all other's tastes and wealth is relaxed, than the Actual Law of Motion (ALM) will most likely be different than the PLM. It appears reasonable to assume that during IPO pre-sale and after trade begins for the share, agents are unable to deduce other's strategies from market prices thus non-REE

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[10] The equilibrium price in (14) exists and is unique since the utility functions from which the supply and demand functions were derived are monotone, and strictly concave over the entire domain. Existence and uniqueness can be proved by one of the fixed-point theorems for smooth, continuous functions, e.g., Debreu (in Theory of Value, 1959) or Kakutani (in McKenzie L., On Equilibrium in Graham's Model of World Trade, Econometrica, Vol. 22, pp. 147-161, 1954)

pricing might prevail. The divergence of the ALM from PLM will last longer if the share is of a new-technology since idiosyncratic risk makes dynamic learning a longer and more complex process. Such divergence will take the shape of a "boom" and "crash" pattern if the return and variability are serially correlated at each phase separately. In the following sub-sections we show that this will be the case for the "boom" phase, while the "crash" is discussed in the next section.

A. Average Growth Rates

In order to evaluate the realized average return as a function of the expected return when agent heterogeneity is unobservable, take the expected value of (14) and find its time differential,

$$\frac{E(\Delta P^*_{t+\Delta t})}{\Delta t} = \frac{\hat{\lambda}}{N}\left(\left(\frac{S_{M,t}}{\delta_M} + \frac{S_{C,t}}{\delta_C}\right)\hat{\mu} + \left(\frac{D_{M,t}}{\delta_M} + \frac{D_{C,t}}{\delta_C}\right)r\right). \tag{16}$$

Since $D_{K,t}$ corresponds to the optimal amount held in shares at each period, we replace it with $D_{K,t} = S_{K,t}\left(\frac{\delta_K}{\hat{\lambda}} - 1\right) - \eta_K\delta_K$, which is a reorganization of (8), being the agent's investment strategy, but now denoted in terms of the amount held in bonds. Replace the bond strategy into (16), multiply by Δt and divide through by P_t to obtain the actual return process

$$\mu^*_{t+\Delta t} \equiv \frac{E(\Delta P^*_{t+\Delta t})}{P_t} = \frac{\hat{\lambda}}{S_t}\left(\left(\frac{S_{M,t}}{\delta_M} + \frac{S_{C,t}}{\delta_C}\right)\hat{\mu}\Delta t + \left(\frac{S_t}{\hat{\lambda}} - \frac{S_{M,t}}{\delta_M} - \frac{S_{C,t}}{\delta_C} - \eta_M - \eta_C\right)r\Delta t\right).$$

Rearrange terms to represent the above in proportional values and obtain

$$\mu^*_{t+\Delta t} = \Psi_t(\hat{\mu} - r)\Delta t + \left(1 - \hat{\lambda}\left(\frac{\eta_M + \eta_C}{S_t}\right)\right)r\Delta t, \tag{17}$$

where $\Psi_t \equiv \left(\frac{\hat{\lambda}}{\delta_M}\beta_{M,t} + \frac{\hat{\lambda}}{\delta_C}\beta_{C,t}\right)$ is a weighted-average factor of asset holdings

and $\beta_{K,t} = \frac{S_{K,t}}{S_t}, \beta_{C,t} = 1 - \beta_{M,t}$. It turns out that $\mu^*_{t+\Delta t}$ is a linear function of the risk-premium over each time interval Δt. In order to satisfy REE, $\mu^*_{t+\Delta t} = \hat{\mu}$ must hold, for which condition $\Psi_t = 1$ and $\eta_M + \eta_C = 0$ are necessary and sufficient conditions as can be deduced from (17). Assuming zero displacements, if $\Psi_t > 1$ holds, the realized return will be an amplification of $\hat{\mu}$ and if $\Psi_t < 1$ it will be dimmed, i.e., a negative mispricing.

B. Volatility

The variance of the actual share price process, (14) is

$$Var(P^*_{t+\Delta t}) = \left(\frac{\hat{\lambda}}{N}\left(\frac{S_{M,t}}{\delta_M} + \frac{S_{C,t}}{\delta_C}\right)\right)^2\hat{\sigma}^2,$$

or, by replacing $\beta_{K,t} = \dfrac{N_{K,t}}{N}$, the instantaneous standard deviation of price change is

$$\sigma^*_{t+\Delta t} \equiv \sqrt{Var(P^*_{t+\Delta t})} = \left(\frac{\hat{\lambda}}{\delta_M} \beta_{M,t} + \frac{\hat{\lambda}}{\delta_C} \beta_{C,t} \right) \hat{\sigma} = \Psi_t \hat{\sigma} \,. \tag{18}$$

That is, the actual price change variability is a zero-constant linear function of the expected variability, with Ψ_t serving as the coefficient. If $\Psi_t = 1$ the ex-post price path will be identical to the ex-ante, satisfying REE; if $\Psi_t > 1$ the realization will be an amplification of the perceived variability and if $\Psi_t < 1$ the realized variability will be a dimmed multiple of $\hat{\sigma}$.

It is important to note that for both moments, $\hat{\mu}$ and $\hat{\sigma}$, a crowding-out effect intensifies the divergence of the realized moments from the perceived ones. This crowding-out has no limit under MDE but it is limited for the CDE equilibrium as presented in sub-section C below. The reason for the crowding-out under MDE is that agents M buy shares from agents C when $\Psi_t > 1$.[11] By doing so, Ψ_t further increases above its previous level and the crowding-out (stochastically) intensifies in time. The higher the share price grows above the PLM price, agents M demand more shares but agents C have less to offer, thus the periodic equilibrium price closes at higher and higher levels, for declining trade volumes.

C. Dynamic Stability

This sub-section presents the technical properties of the dynamic evolution of the boom phase only, while reversion to REE is discussed in the next section. In general, dynamic stability properties of (the expected value of) a process like (14) are analyzed by adding its homogenous and particular solutions for a given seed, as it is a first order linear difference equation of the type $P_t = P_{t-1} A + B$. Specifying the initial condition P_0 (the IPO price) determines the solution sequence $P_0, P_1, P_2, P_3, \dots$, where each term is found by $P_{t+1} = P_t A + B$ $t = 1,2,3\dots$. By a Theorem of linear first order difference equations, the solution for the sequence can be expressed in terms of P_0, A and B in the following way,

$$P_t = A^t P_0 + B \frac{1 - A^t}{1 - A} \quad \text{for } A \neq 1 \quad t = 0,1,2,3,\dots$$
$$P_t = P_0 + Bt \qquad\qquad \text{for } A = 1 \quad t = 0,1,2,3,\dots$$

where, if $-1 < A < 1$ the solution sequence converges to $\overline{P} = \dfrac{B}{1 - A}$, otherwise it diverges, unless $P_t = P_0$. Rearranging (14) to conform with the above layout yields

$$E(P^*_t) = P_0 \Psi_0 + \frac{\hat{\lambda}}{N} \left(\left(\frac{D^*_{M,0}}{\delta_M} + \frac{D^*_{C,0}}{\delta_C} \right) + \eta_M + \eta_C \right),$$

---

[11] There is a secondary crowding-out effect that persists even when $\Psi_t = 1$, yet with reasonable risk aversion parameters and equal initial wealth allocation it requires 10-20 years for that effect to mature. As noted above, our horizon for post-IPO bubbles is much shorter.

thus

$$A = \Psi_0 = \left(\beta_{M,0} \frac{\hat{\lambda}}{\delta_M} + \beta_{C,0} \frac{\hat{\lambda}}{\delta_C} \right),$$

$$B = \frac{\hat{\lambda}}{N} \left(\frac{D_{M,0}^*}{\delta_M} + \frac{D_{C,0}^*}{\delta_C} + \eta_M + \eta_C \right).$$

This model thus maintains three equilibria sets, depending on the IPO allocation between agents C and M: REE equilibria (no bubbles), Momentum Dominance Equilibria (MDE, positive bubbles) and Contrarian Dominance Equilibria (CDE, negative mispricing).

1) If conditions $\Psi_0 = 1$, and $\eta_M + \eta_C = 0$ hold at IPO date, the expected ALM grows at the expected value of the PLM, resulting in REE. The REE price path can be constructed at a continuum of agents' wealth and taste combinations, all maintain ALM=PLM, but each with a different trade level. The analysis of trade is not discussed in this paper.

2) Momentum Dominance Equilibria will evolve if inequality $\Psi_0 > 1$ holds (for

$\eta_M + \eta_C = 0$) at IPO date, generating positive bubbles whereby $\left(\begin{array}{c} \mu_{t+\Delta t}^* > \hat{\mu} \\ \sigma_{t+\Delta t}^* > \hat{\sigma} \end{array} \right)$. Should

$\eta_M + \eta_C \neq 0$, average return will change with the inequality sign. There is no upper limit to the potential divergence of the ALM from the PLM.

3) Contrarian Dominance Equilibria will emerge if $\Psi_0 < 1$ holds at the IPO,

generating negative mispricing with $\left(\begin{array}{c} \mu_{t+\Delta t}^* < \hat{\mu} \\ \sigma_{t+\Delta t}^* < \hat{\sigma} \end{array} \right)$ if $\eta_M + \eta_C = 0$. As in the previous case,

if $\eta_M + \eta_C \neq 0$, the average return will increase/decline with the inequality sign, being a displacement to the price kernel. Unlike the MDE case however, here the ALM will converge in finite time to a steady state level given by the $t{=}0$ PLM

$$E(\overline{P}_t) = \frac{\hat{\lambda}}{N} \left(\frac{W_{M,0}^*}{\delta_M} + \frac{W_{C,0}^*}{\delta_C} + \eta_M + \eta_C \right).$$

IV. Analysis of Equilibria and Crashes
A. The Benchmark - REE Equilibria
As a benchmark for discussing bubbles, it appears advantageous to analyze the conditions underlying REE equilibria. In order to satisfy REE, ex-ante estimates must hold also ex-post, i.e., both conditions

$$\left(\begin{array}{c} \Psi_0 = 1 \\ \hat{\lambda} \left(\dfrac{\eta_M + \eta_C}{S_0} \right) = 0 \end{array} \right) \tag{19}$$

at (17) and (18) must hold. The first condition, $\Psi_0 = \left(\dfrac{\hat{\lambda}}{\delta_M} \beta_{M,0} + \dfrac{\hat{\lambda}}{\delta_C} \beta_{C,0} \right) = 1$ can

be solved for β_M since $\beta_C = 1 - \beta_M$. In this case the proportion of shares agents M hold out of total shares outstanding is a constant satisfying

$$\beta_M^{REE} = \frac{1/\hat{\lambda} - 1/\delta_C}{1/\delta_M - 1/\delta_C}, \tag{20}$$

which guaranties $\left(\sigma_{t+\Delta t}^* = \hat{\sigma} \right)$. The second condition requires zero-sum displacement factors and, in combination with the first condition, assures $\left(\mu_{t+\Delta t}^* = \hat{\mu} \right)$. Assuming conditions (19) hold, equation (20) shows that there is a continuum of risk-aversion combinations that satisfy REE. This functional relationship between risk aversions and the market price for risk is graphically presented in Figure 2. One can observe that the REE line serves as a border-line that distinguishes between the MDE and CDE equilibria types.

Figure 2
Rational Expectations Equilibrium For
Varying Levels of Risk Aversion and a Given λ
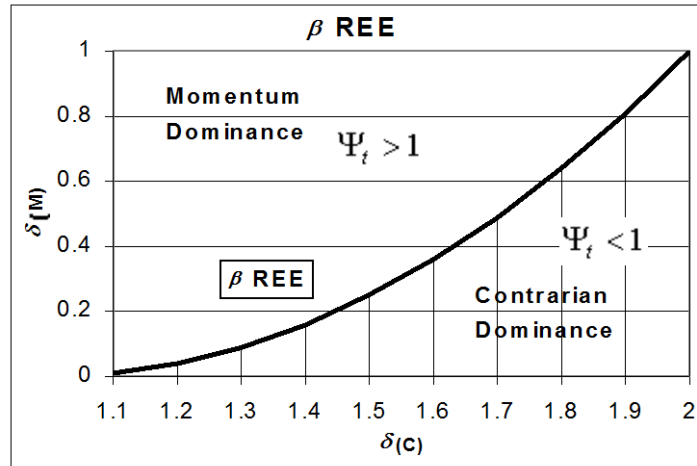
For a given β_M^{REE} there is a continuum of attainable REE equilibria that will prevail for given risk aversions and market price for risk, $\hat{\lambda}$ (here $\hat{\lambda}=1$). Any combination between risk aversions and a given $\hat{\lambda}$ that lies in the area above the β_M^{REE} (i.e., $\Psi_0 = 1$) line is a (temporary) Momentum Dominance Equilibrium (MDE), while below this line the (temporary) Contrarian Dominance Equilibrium (CDE) will prevail.

From $\Psi_0 = \left(\dfrac{\hat{\lambda}}{\delta_M} \beta_{M,0} + \dfrac{\hat{\lambda}}{\delta_C} \beta_{C,0} \right)$ one can see that if agents who buy shares at IPO are more risk-averse Ψ_0 will decline and if less risk-averse agents participate, the value of Ψ_0 increases.

B. Bubbles - Discussion and Illustration

By definition, bubbles are temporary phenomena throughout which prices diverge from REE. In this model it is sufficient to assume incomplete information about agent's tastes and wealth in order to obtain bubbles. It appears that IPO of new-technologies are best candidates to exhibit bubbles since buyers do not observe each other's tastes and wealth at IPO date. After the IPO agents can collect market data, deduce the moments that characterize the stochastic process and revise their strategy based on that information. With new-technology shares however the market data is noisy and might require a long sample, i.e., investment in new-technology assets imply the undertaking of uninsurable risk at least for some period post-IPO. IPOs of companies with existing technologies should not exhibit bubbles since agents can hedge the IPO share by holding assets who's covariability with the IPO asset is non-zero. As long as contingent claims for the new technology do not exist, the only way agents can insure the wealth they hold at the new-technology is by applying dynamic asset allocation strategies. The above Momentum and Contrarian strategies are such optimal strategies as they imply replication of option.

Figure 3a illustrates the average growth rates of the three generic possibilities (REE, MDE and CDE) and Figure 3b presents an REE stochastic sample path, together with an amplification of it under MDE and a dimming version of the same under CDE.
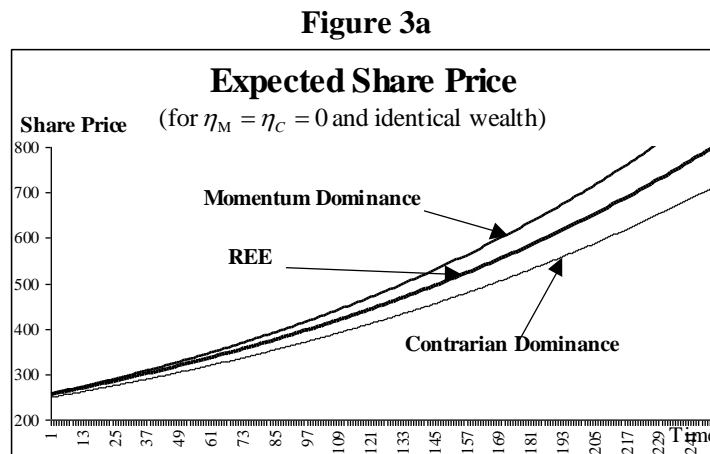
Figure 3a

Expected Share Price

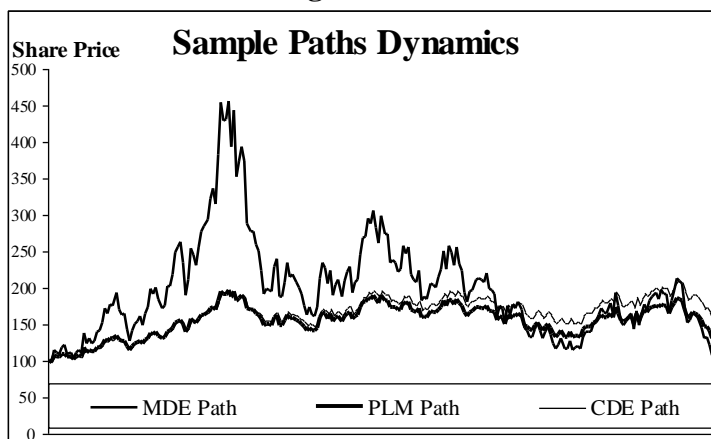(for $\eta_M = \eta_C = 0$ and identical wealth)

Figure 3b

Figure 3a presents expected and Figure 3b sample-paths of the three possible share price equilibria under marginal trade aggregation. In Figure 3b the three paths use the same vector of dz_t, thus differences result from agent heterogeneity. If agents who apply a Momentum strategy dominate (MDE) the equilibrium path will be an amplification of the PLM. If Contrarian agents dominate, the realized path will be a deemed version of the PLM.

C. Reversion of Bubbles to REE - Not Necessarily a "Crash"

The reason for a "crash" or, more generally, reversion of the ALM to REE is intentionally not prescribed in this model in a formal manner. We focused on the reasoning of a divergence of ALM from PLM, which implies divergence from the strict formal definition of REE. Under this definition, RE assumes agents hold a complete knowledge about the economy - its structure (institutions, other agents, markets and changes in those) and value of parameters in the economy. Alternatively, agents must at least agree on a model of the economy and its parameters. For a stochastic model, agreement on the distribution of shocks is also required. Empirical economists who test for rationality use past data in order to estimate parameters of a model, or the economy's future state, implying that the experts themselves lack complete knowledge. It thus appear reasonable to assume that agents conduct procedures as if they were econometricians, which means some form of "bounded rationality", as suggested by Sargent (1993).

If diversion of the ALM from PLM is due to incomplete information, as described in the model detailed above, the question is what forces govern a reversion to REE? There are two types of "add-on" models that can be supplemented to our model. The first one stems from the classic view of RE theory. It postulates reversion to REE through agent screening, a process under which agents who fail to accurately predict the PLM loose their wealth to agents who make accurate predictions and thus are driven out of the market in finite time. Since the accurate prediction is the RE solution, the RE pricing will prevail (see Sandroni, 2000 for a comprehensive discussion and proofs). Note that this view implies that an economy with non price-taking agents end-up with temporary deviation from REE, though not necessarily in a boom and crash pattern. If our "boom" reverts to REE based on that notion, the reversion need not be sudden ("crash").

Second type of models comprises the dynamic learning approach. In general, these consider economic agents as econometricians who apply statistical tools on past

data and use the results to establish optimal future decisions. In that sense, this approach set a solid ground for empirical estimation of the above model. In most cases, dynamic learning models use Recursive Least Squares, whereby agents are assumed to run regressions on historic data and estimate the model parameters, testable ex-post by ARMA type processes.[12] If the reversion to REE of post-IPO mispricing is based on that approach, than by the homogeneous information assumption all agents of a given type should apply the same rule once data become available. Since data is available to all simultaneously, a prompt adjustment, i.e., a crash, is unavoidable.

V. Summary

The model proposed above presents a simple technique to calculate asset prices, drawing elements from Merton (1971) and standard microeconomic supply/demand equilibrium. Its major advantages are its ability to specify a wide spectrum of dynamic equilibria; defining a many-to-one correspondence between measures or risk aversion and investment strategies and allowing empirical tests for temporary mispricing. It turns out that in-spite of homogenous information and rationality, there are infinite combinations of tastes (risk aversion) that result in the single REE path. In addition, we define two sets of temporary equilibria other than the REE. One is a set of Momentum Dominance Equilibria (MDE) in which variability and drift of a PLM are linearly *amplified*. The second is a set of Contrarian Dominance Equilibria (CDE) where the ALM exhibits linear *dimmed* variability and drift vs. the PLM.

We achieve this variety of equilibria since we add a new degree of freedom to asset pricing, being marginal supply and demand functions for shares. We derive the marginal functions from investment strategies that are based on the agent's utility function. The marginal supply/demand functions look like regular microeconomic ones, and in particular they are twice differentiable and strictly convex over the entire domain. These functions may as well serve in the empirical study of price formation being normative bid and ask aggregate functions.

The model is essentially a model of bubbles, while the reversion to REE may be gradual, if some agents who accurately predict the PLM drive the others out of the market. Or, it can end in a "crash" when information from post-IPO market prices makes it clear to all shareholders that the ALM has been constructed based on moments higher than those at the PLM.

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[12] Evans and Honkapohja, 2001 provide a comprehensive survey and detailed expositions of the subject from different perspectives.

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<u>Appendix 1</u>
Table 1
Risk Aversion Properties of a HARA Type Utility Function

Function/ Measure of Risk Aversion	Values Obtained for Specific Conditions	Investor Risk Attitude
$A(W) = \dfrac{1}{\dfrac{W}{1-\gamma} + \eta} > 0$	Implies $\eta > W/(1-\gamma)$ if $\gamma > 1$	Risk Averse
Absolute Risk Aversion		
$A'(W) = \dfrac{-1}{(1-\gamma)\left(\dfrac{W}{1-\gamma} + \eta\right)^2}$	< 0 for $-\infty < \gamma < 1$ $= 0$ for $\gamma = +\infty$	DARA CARA
Relative Risk Aversion		
$R'(W) = \dfrac{\eta}{\left(\dfrac{W}{1-\gamma} + \eta\right)^2}$	< 0 for $\eta < 0, -\infty < \gamma < 1$ $= 0$ for $\eta = 0$	DRRA CRRA

<u>Source:</u> Merton (1971).

Table 1 shows that the HARA type utility function can produce absolute or relative measure of risk aversion, either can be decreasing, increasing or constant. We ignore increasing constant and relative risk aversions.

<center>Appendix 2</center>

Proof of Proposition 1:

A marginal demand function for shares must maintain a negative slope for (10),

$$\frac{\partial P^d_{t+\Delta t}}{\partial(\Delta N_{C,t+\Delta t})} = -\frac{\frac{\hat{\lambda}}{\delta_C}\left(\tilde{D}_{C,t+\Delta t} + \eta_C \delta_C\right)}{\left(N_{C,t}\left(1 - \frac{\hat{\lambda}}{\delta_C}\right) + \Delta N_{C,t+\Delta t}\right)^2} < 0. \tag{A1}$$

Since the market price for risk $\hat{\lambda}$ is ex-ante positive and δ_C must be strictly positive to assure risk aversion, than the numerator must be positive, which implies $\eta_C > -\tilde{D}_{C,t+\Delta t}/\delta_C$. By restricting η_C, we implicitly limit δ_C as well. Solving the asset allocation problem (8), for $\frac{\hat{\lambda}}{\delta_C}$ and replacing $\hat{W}_{C,t+\Delta t} = \hat{S}_{C,t+\Delta t} + \tilde{D}_{C,t+\Delta t}$ and $\alpha_{C,t+\Delta t}\hat{W}_{C,t+\Delta t} = \hat{S}_{C,t+\Delta t}$, we get

$$\frac{\hat{\lambda}}{\delta_C} = \frac{\hat{S}_{C,t+\Delta t}}{\hat{S}_{C,t+\Delta t} + \tilde{D}_{C,t+\Delta t} + \eta_C \delta_C} < 1. \tag{A2}$$

Thus, $\tilde{D}_{C,t+\Delta t} + \eta_C \delta_C > 0$ imply $\delta_C > \hat{\lambda}$. Convexity of the marginal demand function for shares will be satisfied if the second derivative is positive,

$$\frac{\partial P^d_{t+\Delta t}}{\partial(dN^2_{C,t+\Delta t})} = \frac{2\frac{\hat{\lambda}}{\delta_C}\left(\tilde{D}_{C,t+\Delta t} + \eta_C \delta_C\right)}{\left(N_{C,t}\left(1 - \frac{\hat{\lambda}}{\delta_C}\right) + \Delta N_{C,t+\Delta t}\right)^3} > 0. \tag{A3}$$

Given the restrictions in (A1) and (A2), (A3) will be satisfied iff the denominator is positive, thus $0 < \hat{\lambda}\frac{N_{C,t}}{N_{C,t+\Delta t}} < \delta_C$.

Q.E.D.

<div align="center">Appendix 3</div>

Proof of Proposition 2:

Upward slope of the payoff schedule requires (12) to satisfy $\dfrac{\partial P_t^s}{\partial (\Delta N_{M,t})} > 0$, i.e.,

$$\frac{\partial P_{t+\Delta t}^s}{\partial (\Delta N_{M,t+\Delta t})} = -\frac{\dfrac{\hat{\lambda}}{\delta_M}\left(\tilde{D}_{M,t+\Delta t} + \eta_M \delta_M\right)}{\left(N_{M,t}\left(1 - \dfrac{\hat{\lambda}}{\delta_M}\right) + \Delta N_{M,t+\Delta t}\right)^2} > 0, \qquad (A4)$$

which implies $\eta_M < -\tilde{D}_{M,t+\Delta t}/\delta_M$. Assuming agents M are in aggregate net lenders, than η_M must be strictly negative. In order to define the conditions on δ_M, solve (8) for $\dfrac{\hat{\lambda}}{\delta_M}$

$$\frac{\hat{\lambda}}{\delta_M} = \frac{S_{M,t+\Delta t}}{S_{M,t+\Delta t} + D_{M,t+\Delta t} + \eta_M \delta_M} > 1. \qquad (A5)$$

Based on $\eta_M < -\tilde{D}_{M,t+\Delta t}/\delta_M$, (A5) imply $\delta_M < \hat{\lambda}$. In order to assess convexity, the derivative of (A4) must be positive,

$$\frac{\partial P_{t+\Delta t}^s}{\partial (\Delta N_{M,t}^2)} = \frac{2\dfrac{\hat{\lambda}}{\delta_M}\left(\tilde{D}_{M,t+\Delta t} + \eta_M \delta_M\right)}{\left(N_{M,t}\left(1 - \dfrac{\hat{\lambda}}{\delta_M}\right) + \Delta N_{M,t+\Delta t}\right)^3} > 0, \qquad (A6)$$

which implies the denominator must be negative, thus the stricter inequality $\delta_M < \hat{\lambda}\dfrac{N_{M,t}}{N_{M,t+\Delta t}}$ must hold. Note that (A6) cannot be zero since the numerator is strictly negative. In addition, by the assumption of risk aversion, δ_M must be positive, thus $0 < \delta_M < \hat{\lambda}\dfrac{N_{M,t}}{N_{M,t+\Delta t}}$.

Q.E.D.